ROBUST PARAMETER ESTIMATOR FOR NON-LINEAR GROWTH CURVES

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THESIS

ROBUST PARAMETER ESTIMATOR FOR
NON-LINEAR GROWTH CURVES

by

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Robust Parameter Estimator For

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by

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The parameter estimation methods considered in this thesis are the weighted Least-Squares and Weighted Huber for some non-linear growth models. The properties of these parameter estimators derived from simulated data by means of (1) weighted and unweighted least-squares and (2) weighted and unweighted Huber robust estimation are compared. The error components of the simulated data are long-tailed and non-normal. The performance of mis-specified models is considered.

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I. INTRODUCTION

The properties of the least-squares estimator for the linear model are well known [Ref. 2]. Among its important properties is that it has minimum variance among the class of linear unbiased estimators. In Chapter II the properties of the least-squares solution for $\widehat{\ell n A}$ in the growth model

$$Y(t_i) = Ae^{-B/t_i} e^{\varepsilon(t_i)} \qquad t_i = 1,2,\ldots,n$$

will be reviewed. Then the estimate $\hat{A}$ and its statistical properties will be derived using the estimator for $\widehat{\ell n A}$ by means of the Taylor series approximation. Thus the emphasis will be on estimation of the "final value" of the growth process.

The least-squares estimator is easy to compute and has desirable properties when the assumption of normality is justified. However, many of the properties of the least-squares estimators are not robust against non-normality assumptions (Sheffé, [3]).

Recently (Huber, [1]) statisticians have shown considerable interest in finding estimators which are robust against non-normality of error assumptions. Examples of non-normal distributions of interest are long-tailed distributions such as a mixture of two normals, and also the double exponential and Cauchy. Several robust methods have been considered such as:

1. Using the median,

2. Jackknifing,

3. Trimmed mean,

4. Huber estimator.

Of these methods only the Huber will be considered in Chapter III. One major difficulty with the Huber estimation method is that the solution process leads to a system of non-linear equations which cannot, in general, be solved analytically. Although some asymptotic properties of the Huber estimator have been derived, the small sample properties must generally be obtained by computer simulation.

Finally, a problem is said to be mis-specified if the data comes from the model

$$Y_1 = g(x,\varepsilon_1) \tag{1-1}$$

but the model used is

$$Y_2 = f(x,\varepsilon_2) \tag{1-2}$$

where the functions f and g are not identical.

Since in practice the true model is not known, it is important to know how good are the fits or estimates obtained by using reasonable alternative models. In Chapter IV the properties of several mis-specified model fits are compared when the true model is

$$Y(t_i) = A \frac{Bt_i}{1 + Bt_i} e^{\varepsilon(t_i)}$$

for A = 1 and $t_i = 1,2,\ldots 20$ and B = 0.0772 using computer simulation. The reason for choosing B = 0.0772 is that for

this value of B, the largest value of E(Y) for the true model is 0.6, i.e. 60% of the final value (A = 1) eventually reached.

II. LEAST-SQUARES ESTIMATION OF ℓnA AND TAYLOR SERIES APPROXIMATION OF A

A. LEAST-SQUARES ESTIMATION OF ℓnA

Consider the model

$$Y = Ae^{-B/t}\, e^{\varepsilon} \tag{2-1}$$

where t is the independent variable, Y is the dependent variable, and ε a random variable.

Let the n pairs of sample observations of Y and t be (t_1, y_1), (t_2, y_2), $\cdots$,(t_n, y_n).

Assume the hypothesis

$$E(\varepsilon_i) = 0 \qquad i = 1, \quad ,n$$

$$E(\varepsilon_i \varepsilon_j) = \begin{cases} 0 & i \neq j \\ \sigma_\varepsilon^2 & i = j \end{cases} \tag{2-2}$$

The distribution of $\ell n\hat{A}$ and the approximate distribution of $\hat{A}$ may be found by taking logarithms of equation (2-1), i.e.

$$\ell n\ Y = \ell n\ A - \beta/t + \varepsilon. \tag{2-3}$$

The properties of $\ell n\hat{A}$ can be derived by considering the linear model

$$Z = \alpha + \beta x + u \tag{2-4}$$

where

$Z = \ell n\ Y$, $\alpha = \ell n\ A$, $\beta = -\beta$, $x = 1/t$, $u = \varepsilon$.

Assume the following hypothesis for the linear model

$$z_i = \alpha + \beta\ x_i + u_i \qquad i = 1,2,\ldots,n$$

$$E(u_i) = 0 \qquad i = 1,\dots,n$$

$$E(u_i u_j) = \begin{cases} 0 & i \neq j \\ \sigma_u^2 & i = j \end{cases} \tag{2-5}$$

where α, β and σ_u^2 are unknown parameters. The principle of least-squares is to choose estimates $\hat{\alpha}$ and $\hat{\beta}$ of α and β such that $\sum_{i=1}^{n} u_i^2$ is minimized.

The estimates $\hat{\alpha}$ and $\hat{\beta}$ are given by

$$\hat{\beta} = \frac{\sum_{i=1}^{n} (x_i-\bar{x})(z_i-\bar{z})}{\sum_{i=1}^{n} (x_i-\bar{x})^2} \tag{2-6}$$

$$\hat{\alpha} = \bar{z} - \hat{\beta}\,\bar{x}$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$$

from these expressions it can be shown that

$$E(\hat{\alpha}) = \alpha$$

$$\mathrm{Var}(\hat{\alpha}) = \frac{\sum_{i=1}^{n} x_i^2}{n\sum_{i=1}^{n} (x_i-\bar{x})^2}\,\sigma_u^2 \,. \tag{2-7}$$

B. TAYLOR SERIES APPROXIMATION FOR THE LEAST-SQUARES ESTIMATOR A

Let X be a random variable and Y = f(X), where it is assumed that the function f can be expanded in a Taylor series about E(X) = μ; that is,

$$f(X) = f(\mu) + (X-\mu)f'(\mu) + O((x-\mu)^2).$$

Neglecting the higher order terms

$$E(f(X)) \simeq E(\ f(\mu) + (X-\mu)f'(\mu)\ )$$
$$\simeq f(\mu)$$
$$\mathrm{Var}(f(X)) \simeq E(\ (f(X) - E(f(X)))^2\ )$$
$$\simeq E(\ (f(X) - f(\mu))^2\ )$$
$$\simeq E(\ (X-\mu)^2\ )\cdot(f'(\mu))^2. \qquad (2\text{-}9)$$

From (2-3)

$$z = \ell n\ Y = \ell n\ A - B/t + \varepsilon$$

and from (2-7)

$$\mathrm{Var}(\widehat{\ell n A}) = \frac{\sum_{i=1}^{n} x_i^2}{n \sum_{i=1}^{n} (x_i - \bar{x})^2}\, \sigma_\varepsilon^2 .$$

Using the relationship $A = e^{\ell n A}$ and the results of (2-8) and (2-9)

$$\mu = E(X) = \ell n A$$
$$E(\hat{A}) \simeq e^{\mu}$$
$$\mathrm{Var}(\hat{A}) \simeq (e^{\ell n A})\cdot \mathrm{Var}(\ell n \hat{A}) \qquad (2\text{-}10)$$

$$\simeq A^2 \frac{\sum_{i=1}^{n} x_i^2}{n \sum_{i=1}^{n} (x_i - \bar{x})^2}\, \sigma_\varepsilon^2 .$$

Equation (2-10) shows that using the least-squares method and the Taylor series approximation the variance of $\hat{A}$ is directly proportional to σ_ε^2.

As an example,

for $t = 1, 2, \ldots, 20$, $A = 1$, and $B = 10.0$

$$\mathrm{Var}(\hat{A}) = 0.0841 \cdot \mathrm{Var}(\varepsilon)$$

III. THE HUBER ESTIMATION METHOD

A. THE HUBER ESTIMATOR WITH SCALE UNKNOWN

Although least-squares gives the best linear (in the observations) estimates of the parameters in (2-3) or (2-4), non-linear estimates may be appropriate when the error terms (ε's) are long-tailed, as is often true in practice. Various principles for deriving estimates are possible. In this thesis an estimator due to Huber [Ref. 1] is utilized.

$$\text{Prob}\{y \le Y \le y + dy\}$$

$$= \rho\left(\frac{y-\theta}{a}\right)\frac{1}{a}\,dy \qquad (3\text{-}1)$$

where $Y = \theta + a\,\varepsilon$

$\rho(z)$ is the density for ε.

The likelihood function for the observations is

$$\prod_{j=1}^{n}\left[\frac{1}{a}\,\rho\left(\frac{y_j-\theta}{a}\right)\right]$$

The log likelihood function

$$L = \sum_{j=1}^{n} \log\left[\rho\left(\frac{y_j-\theta}{a}\right)\frac{1}{a}\right]$$

$$= \sum_{j=1}^{n}\left[\log\rho\left(\frac{y_j-\theta}{a}\right) - \log a\right].$$

Consider

$$\frac{\partial L}{\partial\theta} = \sum_{j=1}^{n}\frac{\rho'\left(\frac{y_j-\theta}{a}\right)}{\rho\left(\frac{y_j-\theta}{a}\right)}\left(\frac{-1}{a}\right)$$

$$\frac{\partial L}{\partial a} = \sum_{j=1}^{n} \left[\frac{\rho'\left(\frac{y_j - \theta}{a}\right)}{\rho\left(\frac{y_j - \theta}{a}\right)} (-1)\left(\frac{y_j - \theta}{a}\right) \right].$$

To find the maximum of L where θ and a are unknown set $\partial L/\partial\theta$ and $\partial L/\partial a$ equal to zero.

$$\sum_{j=1}^{n} \frac{\rho'\left(\frac{y_j - \theta}{a}\right)}{\rho\left(\frac{y_j - \theta}{a}\right)} = 0 \tag{3-2}$$

$$\frac{1}{n} \sum_{j=1}^{n} \left(\frac{y_j - \theta}{a}\right)\cdot(-1) \frac{\rho'\left(\frac{y_j - \theta}{a}\right)}{\rho\left(\frac{y_j - \theta}{a}\right)} = 1 .$$

In the Huber method with scale unknown, let

$$-\frac{\rho'(z)}{\rho(z)} = \psi(z) \tag{3-3}$$

for some function ψ to be chosen.

The equations can be written in the form

$$\sum_{j=1}^{n} \psi\left(\frac{y_j - \theta}{a}\right) = 0$$

$$\frac{1}{n} \sum_{j=1}^{n} \psi\left(\frac{y_j - \theta}{a}\right)\left(\frac{y_j - \theta}{a}\right) = 1 \tag{3-4}$$

Example 1:

Let $\rho(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

where ρ is the standard normal distribution. Then

$$\rho'(z) = -z \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

The likelihood equations can be written in the form

$$\sum_{j=1}^{n} - \left(\frac{y_j - \theta}{a} \right) = 0$$

$$\frac{1}{n} \sum_{j=1}^{n} \left(\frac{y_j - \theta}{a} \right)\left(\frac{y_j - \theta}{a} \right) = 1$$

which simplifies to

$$\hat{\theta} = \frac{1}{n} \sum_{j=1}^{n} y_j \quad \text{and}$$

$$\hat{a} = \frac{1}{n} \sum_{j=1}^{n} (y_j - \hat{\theta})^2 .$$

These estimates are obtained by using

$$\psi(z) = - \frac{\rho'(z)}{\rho\ (z)} = z . \tag{3-5}$$

Thus the estimates with the normal distribution assumption come from the use of $\psi(z) = z$.

<u>Example 2</u>:

Let ρ be Cauchy. Then

$$\rho(z) = \frac{1}{\pi[1 + z^2]} \quad \text{and}$$

$$\psi(z) = - \frac{\rho'(z)}{\rho\ (z)} = \frac{2z}{1 + z^2} .$$

Forms of ψ that are generally used

<u>HUBER M</u>

$$\psi(z) = \begin{cases} -ka & z \leq -ka \\ z & -ka < z < ka \\ +ka & z \geq ka \end{cases} \tag{3-6}$$

SINE

$$\psi(z) = \begin{cases} \sin\left(\frac{kz}{a}\right) & |z| \leq k\pi \\ 0 & \text{otherwise.} \end{cases} \qquad (3\text{-}7)$$

TUKEY

$$\psi(z) = \begin{cases} kz(1 - kz/a) & |z| < ka \\ 0 & |z| > ka \end{cases} \qquad (3\text{-}8)$$

The parameter a must be selected with reference to a scale parameter for ε.

B. ROBUST ESTIMATION OF PARAMETERS

Consider the linear model

$$y_j = \alpha + \beta(x_j - \bar{x}) + \varepsilon_j \qquad j = 1,2,\ldots,n$$

$$E(\varepsilon_j) = 0$$

$$E(\varepsilon_i \varepsilon_j) = 0 \quad i \neq j$$

$$E(\varepsilon_i^2) = \sigma_\varepsilon^2.$$

Let the scale factor be a. The likelihood function for the observations is

$$L = \prod_{j=1}^{n} \rho\left(\frac{y_j - \alpha - \beta(x_j - \bar{x})}{a}\right) \cdot \frac{1}{a}$$

where $\bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j$.

Differentiating with respect to α, β, and a and equating each derivative to zero

$$\frac{\partial L}{\partial \alpha} = \sum_{j=1}^{n} \left(\frac{\rho'}{\rho} \right) \cdot \left(-\frac{1}{a} \right) = 0$$

$$\frac{\partial L}{\partial \beta} = \sum_{j=1}^{n} \left(\frac{\rho'}{\rho} \right) \cdot (-1) \cdot \left(\frac{x_j - \bar{x}}{a} \right) = 0 \qquad (3\text{-}9)$$

$$\frac{\partial L}{\partial a} = \sum_{j=1}^{n} \left[(-1) \left(\frac{\rho'}{\rho} \right) \cdot \left(\frac{y_j - \alpha - \beta(x_j - \bar{x})}{a^2} \right) - \frac{1}{a} \right] = 0$$

Let

$$\psi(z) = - \frac{\rho'(z)}{\rho(z)} \text{ and}$$

$$r_j = y_j - \alpha - \beta\,(x_j - \bar{x})$$

then the equations (3-9) can be written in the form

$$\sum_{j=1}^{n} \psi\left(\frac{r_j}{a} \right) = 0$$

$$\sum_{j=1}^{n} \psi\left(\frac{r_j}{a} \right) \cdot (x_j - \bar{x}) = 0 \qquad (3\text{-}10)$$

$$\sum_{j=1}^{n} \left[\psi\left(\frac{r_j}{a} \right) \left(\frac{r_j}{a^2} \right) - \frac{1}{a} \right] = 0.$$

A method of iteration for solving the system of non-linear equations (3-10) will now be described. First, start with initial values

$\alpha(0)$, $\beta(0)$, $a(0)$.

Re-arrange equations (3-10): the first

$$\sum_{j=1}^{n} \psi\left(\frac{r_j}{a} \right) = 0 \quad \text{becomes} \quad \sum_{j=1}^{n} \frac{\psi\left(\frac{r_j}{a} \right)}{\left(\frac{r_j}{a} \right)} \cdot r_j = 0$$

and the second equation

$$\sum_{j=1}^{n} \psi\left(\frac{r_j}{a}\right)(x_j - \bar{x}) = 0 \text{ becomes } \sum_{j=1}^{n} \frac{\psi\left(\frac{r_j}{a}\right)}{\left(\frac{r_j}{a}\right)} (x_j - \bar{x})\cdot r_j = 0$$

and the third equation

$$\sum_{j=1}^{n} \left[\psi\left(\frac{r_j}{a}\right)\left(\frac{r_j}{a^2}\right) - \frac{1}{a}\right] = 0 \text{ is now } a = \frac{1}{n} \sum_{j=1}^{n} \psi\left(\frac{r_j}{a}\right)\cdot r_j .$$

$$\sum_{j=1}^{n} \frac{\psi\left(\frac{r_j(k)}{a(k)}\right)}{\left(\frac{r_j(k)}{a(k)}\right)} [y_j - \alpha(k+1) - \beta(k+1)(x_j - \bar{x})] = 0 \tag{3-11}$$

$$\sum_{j=1}^{n} \frac{\psi\left(\frac{r_j(k)}{a(k)}\right)}{\left(\frac{r_j(k)}{a(k)}\right)} (x_j - \bar{x})[y_j - \alpha(k+1) - \beta(k+1)(x_j - \bar{x})] = 0 \tag{3-12}$$

$$a(k+1) = \frac{1}{n} \sum_{j=1}^{n} \psi\left(\frac{r_j(k+1}{a(k)}\right) r_j(k+1). \tag{3-13}$$

From equations (3-11) and (3-12) expressions for $\alpha(k+1)$ and $\beta(k+1)$ may be obtained. These can be substituted into equation (3-13) to obtain $a(k+1)$. With a reasonably good guess for $\alpha(0)$, $\beta(0)$ and $a(0)$ a few iterations should provide sufficiently accurate solutions of the system of equations (3-10).

Initial values for the parameters $\alpha(0)$, $\beta(0)$ and $a(0)$ may be obtained by least squares regression.

C. PROPERTIES OF THE HUBER ESTIMATOR

Consider the model $Y = Ae^{-B/t}e^{\varepsilon}$.

Simulations were carried out with the above model for $A = 1$, $B = 10$, $t = 1,2,3,\ldots,20$. To investigate the properties

of the Huber estimation when errors are long-tailed a "wild shot" distribution with density function $\rho_\varepsilon(x)$ given by

$$\rho_\varepsilon(x) = \frac{1}{\sqrt{2\pi}\,k} e^{-x^2/2k^2}\cdot(q) + \frac{1}{\sqrt{2\pi}\,k\cdot 3} e^{-x^2/2k^2\cdot 9}\cdot(1-q)$$

was used. Four values of q were used: q = 0.9, 0.825, 0.75, 0.6. The values of k were then chosen so that the distribution $\rho_\varepsilon(x)$ had a variance in the range 0.05 to 2.0. For each set of parameter values, the simulation was replicated 100 times to obtain 100 values of $\hat{A}$ from which the sample mean of $\hat{A}$, and the sample variance were obtained. The sample distribution of $\hat{A}$ was tested for normality by means of a plotting subroutine described in Appendix A.

The properties of the Huber estimate $\hat{A}$ obtained from the simulation are

1. The Huber estimate $\hat{A}$ has variance which seems proportional to Var(ε). Examination of Table I, Appendix B, indicates that the variance of the Huber estimator is smaller than the variance of the least-squares estimator.

2. The Huber estimate $\hat{A}$ has an approximate normal distribution for $0 \leq \text{Var}(\varepsilon) \leq 0.5$.

3. For the "wild shot" distribution, which is symmetric, the Huber estimator seems unbiased.

From Table I, $\hat{\text{Var}}(\hat{A}) = 7.23 \times 10^{-3}$, $\bar{\hat{A}} = 1.0056$.

For the hypothesis $E(\hat{A}) = 1.0$

$$\frac{\bar{\hat{A}}-1.0}{\sqrt{7.23\text{x}10^{-3}/100}}$$

should have an approximate t-distribution with 100 degrees of freedom.

The t-test at 0.10 level of significance is

$$t_{0.05} \leq \frac{\bar{\hat{A}} - 1.0}{\sqrt{7.23\text{x}10^{-3}/100}} \leq t_{0.95}$$

which may be rewritten as

$$1.0 - t_{0.05} \sqrt{7.23\text{x}10^{-3}/100} \leq \bar{\hat{A}} \leq 1 + t_{0.95} \sqrt{7.23\text{x}10^{-3}/100}$$

$$t_{0.95}\sqrt{7.23\text{x}10^{-3}/100} = 0.0146$$

$$0.9854 \leq \bar{\hat{A}} \leq 1.0146.$$

The simulated value of $\bar{\hat{A}}$ of 1.0056 is within the confidence interval identified above.

4. A good fit for the variance $\hat{A}$ against Var(ε) is given by

$$\hat{\text{Var}}(\hat{A}) = D\ \text{Var}(\varepsilon) + \delta \cdot [\text{Var}(\varepsilon)]^2$$

where δ is a random variable which is approximately normal for $0 \leq \text{Var}(\varepsilon) \leq 0.5$ with $E(\delta) = 0$. This is shown in Fig. 5 where $\text{Var(A)}/[\text{Var}(\varepsilon)]^2$ is plotted against $1/\text{Var}(\varepsilon)$.

5. For values of q between 0.6 and 0.9, the approximate relationship between $\text{Var}(\hat{A})$ and Var(ε) is

$$\text{Var}(\hat{A}) \simeq 0.066\ \text{Var}(\varepsilon).$$

This relationship is obtained from Table III, Appendix B, by taking the arithmetic mean for q = 0.6, 0.75, 0.825 and 0.9. The reason for using the arithmetic mean of

the four slopes is that they are quite close to one another (as compared to the spread), and the number of replications (100) was not large enough to obtain a more accurate estimate. In Chapter II the relationship for least-squares obtained from theory was

$$\mathrm{Var}(\hat{A}) = 0.084\ \mathrm{Var}(\varepsilon).$$

Thus the Huber estimator has about 80 percent the variance of the least-squares model.

6. For assumptions on the distribution of ε which represent a severe departure from normality, such as the Cauchy, examination of Table III.2 shows that for the four values of the scale parameter used, the largest value of the variance of the Huber method for A is about 1 while that for the least-squares has a value of up to 10^7.

IV. MISSPECIFICATION PROBLEM

A. LEAST-SQUARES METHOD

The theoretical values of the misspecification function problem when the fit is linear or log linear can be obtained fairly easily. First, assume that $x_1, x_2, \ldots, n_x$ are i, i,d with $E(x) = 0$ and that the function $f(x_1, x_2, \ldots, x_n)$ can be expanded in a Taylor series about the point $(\mu_1, \ldots, \mu_n)^T$.

Let $f_j(x_1, \ldots, x_n) = \frac{\partial f}{\partial x_j}(x_1, \ldots, x_n)$.

Then

$$f(x_1, \ldots, x_n) \simeq f(\mu_1, \ldots, \mu_n) + \sum_{j=1}^{n} (x_j - \mu_j) f_j(\mu_1, \ldots, \mu_n)$$

thus

$$E[f(x_1, \ldots, x_n)] \simeq f(\mu_1, \ldots, \mu_n)$$

and

$$\begin{aligned} \mathrm{Var}(f) &\simeq E\Big[\Big(\sum_{j=1}^{n} (x_j - \mu_j) f_j(\mu_1, \ldots, \mu_n)\Big)^2\Big] \\ &\simeq \sum_{j=1}^{n} [f_j(\mu_1, \ldots, \mu_n)]^2 \cdot E[(x_j - \mu_j)^2] \\ &\simeq \sum_{j=1}^{n} [f_j(\mu_1, \ldots, \mu_n)]^2 \cdot \mathrm{Var}(x_j) \\ &\simeq \mathrm{Var}(x) \cdot \sum_{j=1}^{n} [f_j(\mu_1, \ldots, \mu_n)]^2. \end{aligned}$$

Now, suppose the observed data is from the model

$$Y_j' = 1.0 \frac{c \cdot j}{1 + c \cdot j} e^{\varepsilon_j'} \quad \text{where } c = 0.0772$$

but the model used to fit it is

$$Y_j = A e^{-B/j} e^{\varepsilon_j}.$$

Then

$$\varepsilon_j = \ell n\ y_j - \ell n\ A - B/j,$$

and least-squares demands minimization of $\sum_{j=1}^{n} \varepsilon_j^2$.

The observed values of the y_j are y_j'. Hence

$$\varepsilon_j = \ell n(1.0\ \frac{c \cdot j}{1+c \cdot j}\ e^{\varepsilon_j'}) - \ell n\ A - B/j$$

$$= \ell n(\ \frac{c \cdot j}{1+c \cdot j}\ ) + \varepsilon_j' - \ell n\ A + B/j.$$

Defining L(A,B) as

$$L(A,B) = \sum_{j=1}^{n} \varepsilon_j^2$$

$$= \sum_{j=1}^{n} [\ell n(\ \frac{c \cdot j}{1+c \cdot j}\ ) + \varepsilon_j' - \ell n\ A + B/j]^2.$$

Differentiating this function with respect to A and B and setting equal to zero results in

$$\frac{\partial L}{\partial A} = -\ \frac{2}{A} \sum_{j=1}^{n} [\ell n(\frac{c \cdot j}{1+c \cdot j}) + \varepsilon_j' - \ell n\ A + B/j] = 0$$

$$\frac{\partial L}{\partial B} = \sum_{j=1}^{n} \frac{1}{j}\ [\ell n(\frac{c \cdot j}{1+c \cdot j}) + \varepsilon_j' - \ell n\ A + B/j] = 0.$$

The least-squares solution for ℓn A is

$$\widehat{\ell n A} = \frac{(\sum_{j=1}^{n} \frac{1}{j^2}) \sum_{j=1}^{n} [\varepsilon_j' + \ell n\ \frac{c \cdot j}{1+c \cdot j}] - [\sum_{j=1}^{n} \frac{1}{j}] \sum_{j=1}^{n} \frac{1}{j}[\varepsilon_j + \ell n \frac{c \cdot j}{1+c \cdot j}]}{n\ (\sum_{j=1}^{n} \frac{1}{j^2}) - (\sum_{j=1}^{n} \frac{1}{j})(\sum_{j=1}^{n} \frac{1}{j})} \tag{4-2}$$

If the ε_j are assumed to be iid with $E(\varepsilon_j) = 0$ and $Var(\varepsilon_j)$ small, then the Taylor series approximation gives

$$E(\ell\hat{n}A) = \frac{\left(\sum_{j=1}^{n} \frac{1}{j^2}\right)\left(\sum_{j=1}^{n} \ell n\frac{c\cdot j}{1+c\cdot j}\right) - \left(\sum_{j=1}^{n} \frac{1}{j}\right)\left(\sum_{j=1}^{n} \frac{1}{j}\ \ell n\frac{c\cdot j}{1+c\cdot j}\right)}{n\left(\sum_{j=1}^{n} \frac{1}{j^2}\right) - \left(\sum_{j=1}^{n} \frac{1}{j}\right)^2} \tag{4-3}$$

For the variance of $\ell\hat{n}A$

$$\frac{\partial f}{\partial X_k}(0) = \frac{\left(\sum_{j=1}^{n} \frac{1}{j^2}\right) - \left(\sum_{j=1}^{n} \frac{1}{j}\right)\cdot\frac{1}{k}}{n\left(\sum_{j=1}^{n} \frac{1}{j^2}\right) - \left(\sum_{j=1}^{n} \frac{1}{j}\right)^2} \tag{4-4}$$

Using (4-4) the variance for $\ell\hat{n}A$ becomes

$$Var(\ell\hat{n}A) \simeq Var(X) \sum_{j=1}^{n} \left(\frac{\partial f}{\partial X_k}(0)\right)^2$$

and (4-5)

$$Var(\hat{A}) \simeq e^{E[\ell\hat{n}A]}\cdot Var(\ell\hat{n}A).$$

1. Comparison of Theoretical and Simulated Results

For observed data from the model

$$Y_j^! = 1.0\ \frac{c\cdot j}{1+c\cdot j}\ e^{\varepsilon_j^!} \qquad c = 0.0772$$

$$j = 1,2,\ldots,n \qquad Var(\varepsilon) = 0.072$$

The theoretical values are

$E(\ell\hat{n}A) = -\ 0.5768$

$E(\hat{A}) = 0.5617$

$Var(\ell\hat{n}A) = 6.055 \times 10^{-3}$.

The observed values for computer simulation are

$\overline{(\ell\hat{n}A)} = -\ 0.580$

$\widehat{Var}(\ell\hat{n}A) = 6.04 \times 10^{-3}$.

Normality plots not included with this thesis indicated that $\ell\hat{n}A$ was distributed approximately normal. Hence

$$\frac{\ln\hat{A} - (-0.5768)}{\sqrt{0.00604/100}}$$

has an approximate t distribution with 100 degrees of freedom.

The confidence interval at the 0.10 significance level is given by

$$t_{0.05} \leq \frac{\overline{\ln\hat{A}} - (0.5768)}{\sqrt{0.00604/100}} \leq t_{0.95}$$

or

$$- 0.5901 \leq \overline{\ln\hat{A}} \leq - 0.5635.$$

The observed value from simulation was $\overline{\ln\hat{A}} = - 0.580$ which is well within this confidence interval.

B. WEIGHTED HUBER ESTIMATOR

1. Model $Y = Ae^{-B/t}e^{\varepsilon}$

Quite often weights $\{w_j\}$ are given to the jth observation in order to reduce bias or the mean square error when it is suspected that there is misspecification error in the model. The equations (3-10) are modified to read

$$\sum_{j=1}^{n} w_j \; \psi\left(\frac{r_j}{a} \right) = 0$$

$$\sum_{j=1}^{n} w_j \; \psi\left(\frac{r_j}{a} \right)(x_j - \bar{x}) = 0$$

$$\sum_{j=1}^{n} w_j\left[\psi\left(\frac{r_j}{a} \right)\left(\frac{r_j}{a^2} \right) - \frac{1}{a} \right] = 0$$

and the method of iteration can again be used to solve these systems of equations as before.

2. <u>Model $Y = Ae^{-B/t^c}e^{\varepsilon}$</u>

Since quite often better results can be obtained by modifying the original model by the introduction of an additional parameter the model $Y = Ae^{-B/t}e^{\varepsilon}$ was modified by to read $Y = Ae^{-B/t^c}e^{\varepsilon}$. The following is a derivation of the weighted Huber and Least Squares estimators for this model.

a. Weighted Huber Derivation

The weighted likelihood function for the observations is

$$\prod_{j=1}^{n}\left[\rho\left(\frac{\ell n y_j - \ell n A + B/t_j^c}{a}\right)\right]^{w_j}\left(\frac{1}{a}\right)^{w_j}.$$

For the purpose of simplifying notation replace ℓnA by A and ℓny_j by y_j. Then the log weighted likelihood function is

$$L = \sum_{j=1}^{n}\left[w_j \,\ell n\rho\left(\frac{y_j - A + B/t_j^c}{a}\right) - w_j \,\ell n a\right].$$

Differentiating L with respect to A, B, c and a and setting equal to zero results in

$$\frac{\partial L}{\partial A} = \sum_{j=1}^{n} w_j \left(\frac{\rho'}{\rho}\right)\left(\frac{-1}{a}\right) = 0$$

$$\frac{\partial L}{\partial A} = \sum_{j=1}^{n} w_j \left(\frac{\rho'}{\rho}\right)\frac{1}{at_j^c} = 0$$

$$\frac{\partial L}{\partial c} = \sum_{j=1}^{n} w_j \left(\frac{\rho'}{\rho}\right)\frac{B}{t_j^c}\left(\frac{-\ell n t_j}{a}\right) = 0$$

$$\frac{\partial L}{\partial a} = \sum_{j=1}^{n}\left[w_j \left(\frac{\rho'}{\rho}\right)\left(\frac{y_j - A + B/t_j^c}{-a^2}\right) - \frac{w_j}{a}\right] = 0.$$

Let $\psi(z) = -\frac{\rho'(z)}{\rho(z)}$.

Then the above equations can be written as

$$p \equiv \sum_{j=1}^{n} \frac{w_j}{t_j^c} \psi\left(\frac{y_j - A + B/t_j^c}{a} \right) = 0$$

$$q \equiv \sum_{j=1}^{n} \frac{w_j}{t_j^c} \psi\left(\frac{y_j - A + B/t_j^c}{a} \right) = 0 \qquad (4\text{-}6)$$

$$r \equiv \sum_{j=1}^{n} \frac{w_j \ \ell n \ t_j}{t_j^c} \psi\left(\frac{y_j - A + B/t_j^c}{a} \right) = 0$$

$$s \equiv a \sum_{j=1}^{n} w_j - \sum_{j=1}^{n} w_j (y_j - A + B/t_j^c) \psi\left(\frac{y_j - A + B/t_j^c}{a} \right) = 0$$

The unknown parameters A, B, c and a are the solutions to the system of non-linear equations (4-6) which can be solved by Newton's method of iteration. To simplify notation let $r_j = y_j - A + B/t_j^c$. Then

$$\frac{\partial p}{\partial A} = \sum_{j=1}^{n} w_j \left(\frac{1}{-a} \right) \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial p}{\partial B} = \sum_{j=1}^{n} w_j \frac{1}{a t_j^c} \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial p}{\partial c} = \sum_{j=1}^{n} w_j \frac{B}{a} \left(\frac{-\ell n \ t_j}{t_j^c} \right) \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial p}{\partial a} = \sum_{j=1}^{n} w_j \frac{B}{t_j^c} \left(\frac{1}{-a^2} \right) \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial q}{\partial A} = \sum_{j=1}^{n} \frac{w_j}{t_j^c} \left(\frac{-1}{a} \right) \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial q}{\partial B} = \sum_{j=1}^{n} \frac{w_j}{t_j^c} \frac{1}{a t_j^c} \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial q}{\partial c} = \sum_{j=1}^{n} \frac{w_j}{t_j^c} \frac{B}{a} \left(\frac{-\ell n\ t_j}{t_j^c} \right) \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial q}{\partial a} = \sum_{j=1}^{n} \frac{w_j}{t_j^c} \frac{r_j}{(-a^2)} \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial r}{\partial A} = \sum_{j=1}^{n} \frac{w_j \ell n t_j}{t_j^c} \left(\frac{-1}{a} \right) \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial r}{\partial B} = \sum_{j=1}^{n} \frac{w_j \ell n t_j}{t_j^c} \frac{1}{a t_j^c} \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial r}{\partial c} = \sum_{j=1}^{n} \frac{w_j \ell n t_j}{t_j^c} \frac{B}{a} \left(\frac{-\ell n\ t_j}{t_j^c} \right) \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial r}{\partial a} = \sum_{j=1}^{n} \frac{w_j \ell n t_j}{t_j^c} \frac{r_j}{(-a^2)} \psi'\left(\frac{r_j}{a} \right)$$

$$\frac{\partial s}{\partial A} = -\sum_{j=1}^{n} \left[w_j (-1) \psi\left(\frac{r_j}{a} \right) + w_j r_j \left(\frac{-1}{a} \right) \psi'\left(\frac{r_j}{a} \right) \right]$$

$$\frac{\partial s}{\partial B} = -\sum_{j=1}^{n} \left[w_j \frac{1}{t_j^c} \psi\left(\frac{r_j}{a} \right) + w_j r_j \frac{1}{a t_j^c} \psi'\left(\frac{r_j}{a} \right) \right]$$

$$\frac{\partial s}{\partial c} = -\sum_{j=1}^{n} \left[w_j B \frac{(-\ell n t_j)}{t_j^c} \psi\left(\frac{r_j}{a} \right) + w_j r_j \frac{B}{a} \frac{(-\ell n t_j)}{t_j^c} \psi'\left(\frac{r_j}{a} \right) \right]$$

$$\frac{\partial s}{\partial a} = \sum_{j=1}^{n} w_j - \sum_{j=1}^{n} \left[w_j r_j \frac{r_j}{(-a^2)} \psi'\left(\frac{r_j}{a} \right) \right]$$

Let $z = (z_1, z_2, z_3, z_4)^T = (p, q, r, s)^T$

$x = (x_1, x_2, x_3, x_4)^T = (A, B, c, a)^T.$

Then the iterations are given by

$$A_{i+1} = A_i + \Delta A_i$$

$$B_{i+1} = B_i + \Delta B_i$$

$$c_{i+1} = c_i + \Delta c_i$$

$$a_{i+1} = a_i + \Delta a_i$$

where the Δ's are the solution of the equation

$$\begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} & \frac{\partial z_1}{\partial x_4} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_2}{\partial x_3} & \frac{\partial z_2}{\partial x_4} \\ \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} & \frac{\partial z_3}{\partial x_4} \\ \frac{\partial z_4}{\partial x_1} & \frac{\partial z_4}{\partial x_2} & \frac{\partial z_4}{\partial x_3} & \frac{\partial z_4}{\partial x_4} \end{bmatrix} \begin{bmatrix} \Delta A_i \\ \Delta B_i \\ \Delta c_i \\ \Delta a_i \end{bmatrix} = - \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$$

b. Weighted Least-Squares Derivation

The weighted least-squares estimator for the model

$$Y = Ae^{-B/t^c} e^{\varepsilon}$$

lead to the normal equations

$$p \equiv \sum_{j=1}^{n} w_j [\ln y_j - \ln A + \frac{B}{t_j^c}] = 0$$

$$q \equiv \sum_{j=1}^{n} w_j \frac{1}{t_j^c} [\ln y_j - \ln A + \frac{B}{t_j^c}] = 0$$

$$r \equiv \sum_{j=1}^{n} w_j \frac{1}{t_j^c} \ell n t_j [\ell n y_j - \ell n A + \frac{B}{t_j^c}] = 0$$

and the iterations are given by

$$c_{i+1} = c_i - \frac{[\sum_{j=1}^{n} \frac{\ell n y_j}{t_j c_i} - (\ell n A)_i \sum_{j=1}^{n} \frac{w_j}{t_j c_i} + B_i \sum_{j=1}^{n} \frac{w_j}{t_j 2c_i}]}{[- B_i \sum_{j=1}^{n} w_j \frac{\ell n t_j}{t_j c_i}]}$$

$$B_{i+1} = B_i - \frac{[\sum_{j=1}^{n} \frac{w_j \ell n y_j}{t_j c_{i+1}} - (\ell n A)_i \sum_{j=1}^{n} \frac{w_j}{t_j c_{i+1}} + B_i \sum_{j=1}^{n} \frac{w_j}{t_j 2c_{i+1}}]}{[\frac{1}{(\ell n A)_i} \sum_{j=1}^{n} (\frac{w_j}{t_j c_{i+1}})^2]}$$

$$(\ell n A)_{i+1} = (\ell n A)_i - \frac{[\sum_{j=1}^{n} \frac{w_j \ell n y_j \ell n t_j}{t_j c_{i+1}} - (\ell n A)_i \sum_{j=1}^{n} \frac{w_j \ell n t_j}{t_j c_{i+1}} + B_{i+1} \sum_{j=1}^{n} \frac{w_j}{t_j 2c}}{[\sum_{j=1}^{n} \frac{w_j \ell n t_j}{t_j c_{i+1}}]}$$

C. COMPUTER SIMULATION OF THE MISSPECIFIED MODELS

The data analyzed was generated from the model $Y = 1.0 \frac{Bt}{1+Bt} e$ where $B = 0.0772$ with $\mathrm{var}(\varepsilon) = 0.1$, 0.2 and 0.3 initially. The value of B was chosen so that the expected values have a range from 0.07 to 0.60. The values for t were from one to twenty. The sample mean $\bar{\hat{A}}$ and sample variance $\mathrm{Var}(\hat{A})$ were calculated for a sample size of 100. The weighting system chosen was $w(j) = 1.0, j, j^{1.5}, j^{2.0}, 0.7^{(20-j)}, 0.85^{(20-j)}$ in order to find weights which show promising results so as to concentrate the analysis on the interesting cases.

1. Numerical Results

Preliminary examination of Tables VI, IX, and X show that

1. For the least-squares fit to $Y = Ae^{-B/t^C}e^{\varepsilon}$ the expected value of $\hat{A}$ is practically independent of the system of weights chosen with $E(\hat{A})$ approximately equal to 1.30, which is quite surprising.

2. For the same model $Y = Ae^{-c/t}e^{\varepsilon}$ the least-squares method and the Huber method have approximately the same expected value and variance for low weights.

3. For heavy weighting systems, the bias of the Huber method is considerably less than the bias of the least-squares method.

4. The iteration for the Huber method with the model $Y = Ae^{-B/t^C}e^{\varepsilon}$ for $var(\varepsilon) = 0.2$ had some peculiar features. Specifically,

a. About eighty to ninety percent of the simulations with different random number seeds did not result in convergence.

b. Examination of Table V appears to show that the system of non-linear equations for the Huber method have multiple solutions.

c. When the initial starting values for the Newton iteration are the convergent values of one of the sequences, but if a different seed is used, divergence of the iteration still usually occurred.

The probable cause of the divergence for the iteration of the Huber method for the mis-specified model is that the var(ε) chosen was too large. One of the difficulties of the Huber method for the model $Y = Ae^{-B/t^{C}}e^{\varepsilon}$ is chosing a suitable set of parameters (4) as the initial starting values for the iteration. Usually it is quite difficult to chose even three suitable initial values.

As a result, it was decided to concentrate further analysis of the mis-specified problem using heaving weighting systems with the Huber method for the model $Y = Ae^{-B/t}e^{\varepsilon}$ as these gave the least bias. Also, as the initial value of var(ε) used was too large to be practical, the values of var(ε) was reduced to var(ε) = 0.005, 0.01, and 0.02. The weighting systems chosen were $w(j) = j^{1.5}, j^{2}, j^{3}, j^{4}, 0.85^{(20-j)}, 0.7^{(20-j)}$.

Examination of Tables VII and VIII shows that the system which weighted the later data values heavily and early data values very lightly still gave the best results. The weighting system given by $w(j) = j^{4}$ has the smallest bias and mean square error although it had an estimator $\hat{A}$ with the largest variance.

2. Individual Curve Fitting Trials

The following curve fitting experiments were performed for the Huber method and the least-squares method for the model $Y = Ae^{-c/t}e^{\varepsilon}$ with data generated from the model Y = 1.0 $(B \cdot t/1+B \cdot t)e^{\varepsilon}$ with parameter values previously mentioned. Two curves were plotted in each figure. The curve which has the

smaller value of y for small values of t is the fitted curve. The second curve is the expected value of T for the data generated. The figures are:

1. 6-13 var(ε) = 0.005

2. 14-21 var(ε) = 0.010.

The unweighted least-squares curves (Figures 6,14) were obviously bad fits. Comparison of the various figures show that in general the heavier weighting systems gave better fitted curves for large values of t and poorer fits for small values of t.

V. DISCUSSION

A. JUSTIFICATION FOR HUBER METHOD

For the currently specified model given by $Y = Ae^{-B/t}e^{\varepsilon}$ with $E(\varepsilon) = 0$, the Huber method has a slightly smaller variance than least-squares for the random variable ε consisting of a mixture of two normals. For $Var(\varepsilon) \leq 0.5$, the variance of the Huber estimator $\hat{A}$ is approximately normal. Thus the Huber estimator of A is a better estimator than least-squares for long-tailed distributions. For non-symmetric distributions of ε the least-squares and Huber methods are comparable (see Table IV).

Another possible objection to the least-squares method for the fit $Y = Ae^{-B/t}e^{\varepsilon}$ is that very often it is not known what the true model is. The least-squares method does not seem robust against a mis-specified model, judging from our experiments. Thus the Huber method might well be preferred to the least-squares method when the true model is not known, judging from the experiments carried out to date.

Another variation of the model $Y = Ae^{-B/t}e^{\varepsilon}$ is $Y = A \cdot e^{-B/t+k}e^{\varepsilon}$. For the data $Y = 1.0\ (Bt/1+Bt)e^{\varepsilon}$ was considered. This model did not appear promising for the following reasons:

1. Comparing the plots for the various methods of fit for $Y = Ae^{-B/t}e^{\varepsilon}$ shows that the fitted curves were too flat as compared to the curve $Y = 1.0\ (Bt/1+Bt)$ for values of t satisfying $12 \leq t \leq 20$. Since the parameter k was introduced

to match more closely the slopes of the fitted curve and the data for values of t between say 15 and 20 as well as the y values, this implies that k would have to be negative. However, the function $y = Ae^{-B/t+k}$ has an infinite discontinuity at $t = -k$. Thus values of $k \leq -1$ are not reasonable for the range of values of t considered above. One way of overcoming this difficulty would be to discard the observations for $0 \leq t \leq 5$, since the future values are less dependent on the distant past observations than those observations which are more recent.

B. AREAS OF FURTHER STUDY

1. Since the exact form of the growth is not known, further investigation in this area. For example, the model used to generate the data in this thesis was $Y = A\cdot\frac{Bt}{1+Bt}\, e^{\varepsilon}$. However, there is no reason to believe that $Y = A(1-e^{-ct})e^{\varepsilon}$ or any similar growth model could not have been used.

2. The range of values for t was from one to twenty (1-20). Other ranges on t need to be investigated.

APPENDIX A: SUBROUTINE NMPLOT

The subroutine NMPLOT was used as an indicator as to whether a set of n observations $x_1, x_2, \ldots, x_n$ come from a normal distribution. To carry out the test, the numbers $x_1, x_2, \ldots, x_n$ are sorted in increasing order say $y_1, y_2, \ldots, y_n$. The sample mean $\bar{y} = \frac{1}{n} \sum_{j=1}^{n} y_i$ and the sample variance $s^2 = \frac{1}{n-1} \sum_{j=1}^{n} (y_j - \bar{y})^2$ are then calculated. Let $z_j = \frac{y_j - \bar{y}}{s}$. If the original set $x_1, \ldots, x_n$ were normal, then for rather large n (at least 50) the set z_j, $j=1,\ldots,n$ should have properties similar to that of the ordered statistics from a standard normal distribution. Let Φ be the cumulative distribution function from the standard normal. Then a plot of j/n versus $\Phi(z_j)$ should lie approximately on a straight line passing through the origin with slope one. To decide whether the sample $x_1, \ldots, x_n$ is approximately normal, a plot obtained from n observations from a standard normal and the two plots compared. The subroutine is part of the computer program for the weighted Huber method for the model $Y = Ae^{-Bt}e^{\varepsilon}$.

$Y = 1.\ e^{-10/t} e^{\varepsilon}$ Wild Shot Distribution $q = 0.9$

	HUBER			LEAST-SQUARES		
$Var(\varepsilon)$	$Var\hat{A}$	$Var(\ell n\hat{A})$	$\bar{\hat{A}}$	$Var(\hat{A})$	$Var(\ell n\hat{A})$	$\bar{\check{A}}$
0.01	7.17×10^{-4}	7.01×10^{-4}	1.0005	7.43×10^{-4}	7.33×10^{-4}	1.0003
0.02	1.43×10^{-3}	1.40×10^{-3}	1.0012	1.50×10^{-3}	1.47×10^{-3}	1.0007
0.05	3.47×10^{-3}	3.51×10^{-3}	0.9930	4.26×10^{-3}	4.21×10^{-3}	0.9927
0.10	7.23×10^{-3}	6.91×10^{-3}	1.0056	7.73×10^{-3}	7.34×10^{-3}	1.0036
0.15	1.01×10^{-2}	1.02×10^{-2}	0.9898	1.29×10^{-2}	1.28×10^{-2}	0.9900
0.20	1.49×10^{-2}	1.40×10^{-2}	1.0102	1.59×10^{-2}	1.47×10^{-2}	1.0073
0.30	2.01×10^{-2}	2.03×10^{-2}	0.9870	2.63×10^{-2}	2.56×10^{-2}	0.9897
0.50	3.34×10^{-2}	3.35×10^{-2}	0.9871	4.50×10^{-2}	4.27×10^{-2}	0.9915

Table I. Comparison of Least-Squares with Huber methods $q = 0.9$.

$Y = Ae^{-B/t}e^{\varepsilon}$ Wild Shot Distribution $q = 0.75$

$Var(\varepsilon)$	HUBER $\widehat{Var}(\hat{A})$	HUBER $\widehat{Var}(\ell n\hat{A})$	HUBER $\overline{\hat{A}}$	LEAST-SQUARES $\widehat{Var}(\hat{A})$	LEAST-SQUARES $\widehat{Var}(\ell n\hat{A})$	LEAST-SQUARES $\overline{\hat{A}}$
0.1	0.00581	0.00563	0.9977	0.00681	0.00656	1.0009
0.2	0.0118	0.0117	1.0033	0.0146	0.0144	1.0010
0.3	0.0254	0.0231	1.0150	0.0345	0.0299	1.032
0.4	0.0197	0.0188	1.0064	0.0269	0.0269	0.9977
0.5	0.0181	0.0178	1.0092	0.0314	0.0299	1.014
0.6	0.0273	0.0283	0.980	0.0422	0.0451	0.972
0.7	0.0489	0.0423	1.046	0.0628	0.0552	1.060
0.8	0.0468	0.0414	1.0033	0.0613	0.0525	1.020
0.9	0.0515	0.0489	1.022	0.0701	0.0648	1.019
1.0	0.0876	0.0723	1.039	0.138	0.0996	1.082
1.1	0.0552	0.0494	1.021	0.0773	0.0739	1.011
1.2	0.0427	0.0412	1.020	0.0810	0.0717	1.035
1.3	0.0588	0.0594	1.979	0.0925	0.0977	0.974
1.4	0.108	0.0832	1.081	0.142	0.110	1.10
1.5	0.0789	0.0685	1.026	0.139	0.112	1.048
1.6	0.106	0.107	1.041	0.196	0.163	1.073
1.7	0.0860	0.0919	0.983	0.135	0.132	0.982
1.8	0.170	0.136	1.023	0.175	0.159	1.006
1.9	0.133	0.107	1.030	0.217	0.167	1.082
2.0	0.173	0.133	1.048	0.249	0.195	1.061

Table II. Comparison of Least-Squares with Huber Method $q = 0.75$.

$$Y = 1.0\ e^{-10/t}\ e^{\varepsilon}$$

Var(ε)	q=0.6 $\hat{Var}(\hat{A})$	q=0.75 $\hat{Var}(\hat{A})$	q=0.825 $\hat{Var}(\hat{A})$	q=0.9 $\hat{Var}(\hat{A})$
0.1	0.0060	0.0058	0.0072	0.0068
0.2	0.0144	0.0118	0.0145	0.0156
0.3	0.0257	0.0254	0.0235	0.0209
0.4	0.0222	0.0197	0.0250	0.0241
0.5	0.0327	0.0181	0.0255	0.0329
0.6	0.0393	0.0273	0.0345	0.0434
0.7	0.0458	0.0489	0.0566	0.0490
0.8	0.0492	0.0468	0.0451	0.0508
0.9	0.0668	0.0515	0.0679	0.0553
1.0	0.0512	0.0876	0.0734	0.0683
1.1	0.0697	0.0552	0.0802	0.0681
1.2	0.0871	0.0427	0.0614	0.0939
1.3	0.0807	0.0588	0.0911	0.0909
1.4	0.0752	0.1080	0.1017	0.1340
1.5	0.1381	0.0789	0.0888	0.1530
1.6	0.1091	0.1060	0.0858	0.1389
1.7	0.0933	0.0860	0.1008	0.1214
1.8	0.0982	0.1700	0.1071	0.1970
1.9	0.1493	0.1330	0.1270	0.2070
2.0	0.1685	0.1730	0.2015	0.1650
Slope	0.0641	0.0593	0.071	0.070

Table III.1. Variance for the Huber Method Wild Shot Distribution.

ρ	$\hat{Var}(\hat{A})$	$\hat{Var}(\ell n\hat{A})$	$\bar{\hat{A}}$	$\hat{Var}(\hat{A})$	$\hat{Var}(\ell n\hat{A})$	$\bar{\hat{A}}$
0.03	3.07×10^{-3}	2.69×10^{-3}	1.0004	6.68×10^{8}	1.58	2.58×10^{3}
0.05	3.57×10^{-3}	3.72×10^{-3}	0.994	4.57×10^{-2}	1.19×10^{-1}	0.974
0.10	1.05×10^{-1}	1.97×10^{-1}	1.025	4.07×10^{3}	1.13	7.38
0.20	7.47×10^{-1}	1.42×10^{-1}	1.15	6.10×10^{7}	2.44	7.9×10^{2}

$Y = Ae^{-B/t}e^{\varepsilon}$

Density Cauchy

Scale parameter ρ

Table III.2. Variances for the Huber method Cauchy Distribution.

ε is mixture of two distributions

Z_1 with probability 0.9

Z_2 with probability 0.1

X is the standard normal

		HUBER		LEAST SQUARES	
	ρ	$\hat{A}$	$\hat{Var}(\hat{A})$	$\hat{A}$	$\hat{Var}(\hat{A})$
Z_1	0.0707	1.020	0.00100	1.021	0.00099
$0.25 + \rho X$	0.1	1.029	0.00147	1.030	0.00147
	0.1414	1.030	0.00218	1.032	0.00215
	0.0707	1.023	0.00102	1.024	0.00101
$Z_2=0.25$	0.10	1.030	0.00142	1.030	0.00141
	0.1414	1.020	0.00203	1.020	0.00201

Table IV. Comparison of Least Squares and Huber Methods for Non-Symmetric Distributions.

Table V. Convergent Sequences for the Huber Method with Misspecified Model.
$Y = Ae^{-B/t^c} e^{\varepsilon}$ with weights w(j) = 1

Iteration	$(\ell nA)_i$	B_i	c_i	a_i
1	0.2000	3.000	0.5000	0.8000
.				
.				
.				
16	-0.1083	3.6682	0.7245	-18.179
17	-0.1086	3.6678	0.7246	-18.072
18	-0.1086	3.6678	0.7246	-17.965
19	-0.1086	3.6678	0.7246	-17.858
15	0.7872	3.1258	0.2924	9.006
16	0.7969	3.1349	0.2911	8.226
17	0.7971	3.1351	0.2910	7.437
18	0.7971	3.1351	0.2911	6.649
19	0.7971	3.1351	0.2911	5.862
20	0.7971	3.1351	0.2911	5.076
.				
.				
.				
15	0.0912	2.6268	0.4214	0.8348
16	0.0912	2.6268	0.4214	0.7991
17	0.0912	2.6268	0.4214	0.7657
18	0.0912	2.6268	0.4214	0.7348
19	0.0912	2.6268	0.4214	0.7064
20	0.0912	2.6268	0.4212	0.6803
.				
.				
.				
15	0.3407	3.2417	0.4544	0.1975
16	0.3346	3.2353	0.4560	0.1929
17	0.3360	3.2367	0.4557	0.1939
18	0.3356	3.2363	0.4558	0.1936
19	0.3357	3.2364	0.4557	0.1937
20	0.3357	3.2364	0.4558	0.1936

$$Y = 1.0 \frac{Bt}{1+Bt} e^{\varepsilon} \qquad b = 0.0772 \qquad q = 0.9$$

$w(j)$	$Var(\varepsilon)$	$\hat{Var}(\hat{A})$	$\bar{\hat{A}}$	M.S.E.
1	0.10	0.00041	0.569	0.186
j	0.10	0.0055	0.681	0.107
$j^{1.5}$	0.10	0.0123	0.697	0.104
j^2	0.10	0.0220	0.740	0.090
0.70^{20-j}	0.10	0.0414	0.761	0.098
0.85^{20-j}	0.10	0.0038	0.655	0.123
1	0.20	0.0154	0.560	0.195
j	0.20	0.0149	0.663	0.128
$j^{1.5}$	0.20	0.0325	0.722	0.110
j^2	0.20	0.068	0.78	0.116
0.70^{20-j}	0.20	0.127	0.839	0.153
0.85^{20-j}	0.20	0.0138	0.674	0.120
1	0.30	0.0209	0.570	0.206
j	0.30	0.0183	0.645	0.144
$j^{1.5}$	0.30	0.0344	0.701	0.124
j^2	0.30	0.066	0.754	0.127
0.70^{20-j}	0.30	0.199	0.852	0.221
0.85^{20-j}	0.30	0.0134	0.663	0.127

Table VI. Weighted Huber Method with Mis-specified Model $Y = Ae^{-c/t}e^{t}$.

$$Y = 1.0 \frac{Bt}{1+Bt} e^{\varepsilon} \qquad B = 0.0772 \qquad q = 0.9$$

$w(j)$	$Var(\varepsilon)$	$\hat{Var}(\hat{A})$	$\bar{\hat{A}}$	M.S.E.
0.70^{20-j}	0.005	0.00333	0.757	0.0629
0.85^{20-j}	0.005	0.00043	0.662	0.115
$j^{1.5}$	0.005	0.00056	0.710	0.0847
j^{2}	0.005	0.00118	0.749	0.0642
0.70^{20-j}	0.01	0.00503	0.756	0.0646
0.85^{20-j}	0.01	0.00087	0.664	0.114
$j^{1.5}$	0.01	0.00121	0.718	0.0807
j^{2}	0.01	0.00154	0.750	0.0640
0.70^{20-j}	0.02	0.0119	0.781	0.0600
0.85^{20-j}	0.02	0.00117	0.659	0.117
$j^{1.5}$	0.02	0.00253	0.707	0.0883
j^{2}	0.02	0.00432	0.742	0.0709

Table VII. Weighted Huber Method with Misspecified Model $Y = Ae^{-c/t} e^{\varepsilon}$.

$Y = 1.0 \frac{Bt}{1+Bt} e^{\varepsilon}$ $B = 0.0772$ $q = 0.9$

w(j)	Var(ε)	$\hat{Var}(\hat{A})$	$\bar{\hat{A}}$	M.S.E.
j^3	0.005	0.0033	0.795	0.0453
j^4	0.005	0.0043	0.835	0.0315
j^3	0.01	0.0035	0.796	0.0451
j^4	0.01	0.0077	0.816	0.0416
j^3	0.02	0.0092	0.790	0.0533
j^4	0.02	0.0179	0.839	0.0438

Table VIII. Weighted Huber Method with Misspecified Model $Y = Ae^{-c/t} e^{\varepsilon}$

$$Y = 1.0 \frac{Bt}{1+Bt} e^{\varepsilon} \qquad B = 0.0772 \qquad q = 0.9$$

$w(j)$	$Var(\varepsilon)$	$\hat{Var}(\hat{A})$	$\bar{\hat{A}}$	M.S.E.
1	0.10	0.00430	1.30	0.094
j	0.10	0.00415	1.33	0.113
$j^{1.5}$	0.10	0.0067	1.30	0.097
j^2	0.10	0.0074	1.29	0.092
0.70^{20-j}	0.10	0.00683	1.26	0.074
0.85^{20-j}	0.10	0.0030	1.30	0.093
1	0.20	0.00896	1.28	0.087
j	0.20	0.0126	1.34	0.128
$j^{1.5}$	0.20	0.0146	1.30	0.105
j^2	0.20	0.0182	1.32	0.121
0.7^{20-j}	0.20	0.0154	1.25	0.078
0.85^{20-j}	0.20	0.0140	1.32	0.116
1	0.30	0.0165	1.28	0.095
j	0.30	0.0130	1.34	0.129
$j^{1.5}$	0.30	0.0179	1.31	0.114
j^2	0.30	0.0268	1.29	0.111
0.70^{20-j}	0.30	0.0361	1.27	0.109
0.85^{20-j}	0.30	0.0139	1.31	0.110

Table IX. Least-Squares Method with Misspecified Model
$Y = Ae^{-D/t^c} e^{\varepsilon}$

$$Y = 1.0\ \frac{Bt}{1+Bt} \qquad B = 0.0772 \qquad q = 0.9$$

$w(j)$	$Var(\varepsilon)$	$\hat{Var}(\hat{A})$	$\bar{\hat{A}}$	M.S.E.
1	0.10	0.00285	0.566	0.191
j	0.10	0.00526	0.654	0.125
$j^{1.5}$	0.10	0.00685	0.673	0.114
j^2	0.10	0.0152	0.738	0.084
0.70^{20-j}	0.10	0.0194	0.755	0.079
0.85^{20-j}	0.10	0.00677	0.652	0.129
1	0.20	0.00517	0.572	0.188
j	0.20	0.0116	0.654	0.131
$j^{1.5}$	0.20	0.0214	0.711	0.105
j^2	0.20	0.0200	0.694	0.114
0.7^{20-j}	0.20	0.0599	0.741	0.127
0.85^{20-j}	0.20	0.0139	0.658	0.131
1	0.30	0.00831	0.570	0.193
j	0.30	0.0165	0.668	0.127
$j^{1.5}$	0.30	0.0181	0.673	0.125
j^2	0.30	0.0429	0.751	0.105
0.70^{20-j}	0.30	0.0597	0.733	0.131
0.85^{20-j}	0.30	0.0230	0.658	0.134

Table X. Least-Squares Method with Misspecified Model
$Y = Ae^{-c/t}\ e^{\varepsilon}$

Figure 1. Comparison of Variances of Huber and Least-Squares Estimator, var(ε) ≤ 0.5.

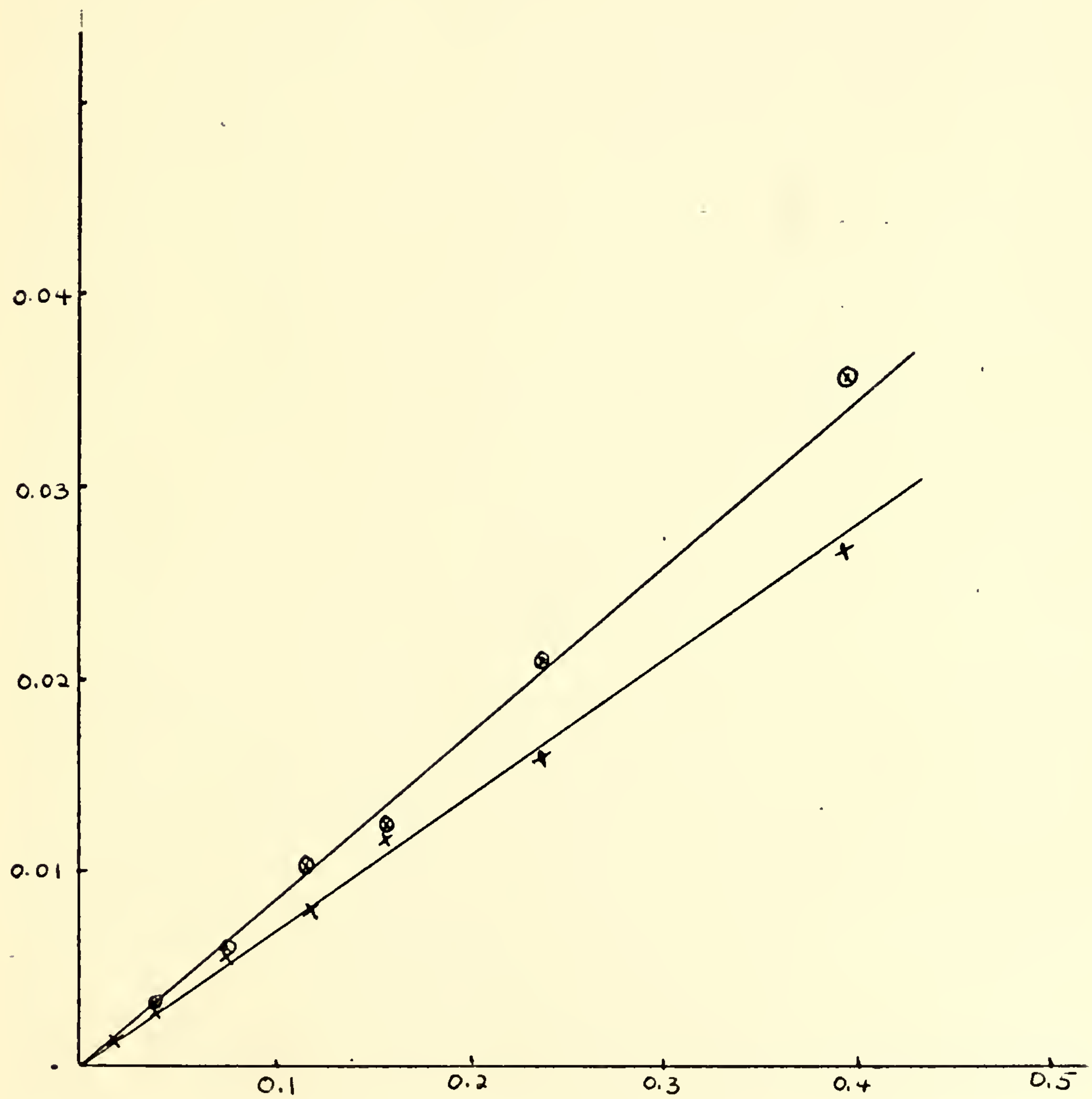


Figure 2. Linear Fit for Variance of Huber Estimator for q = 0.9.

Figure 3. Linear Fit for Variance of Huber Estimator for $q = 0.825$.

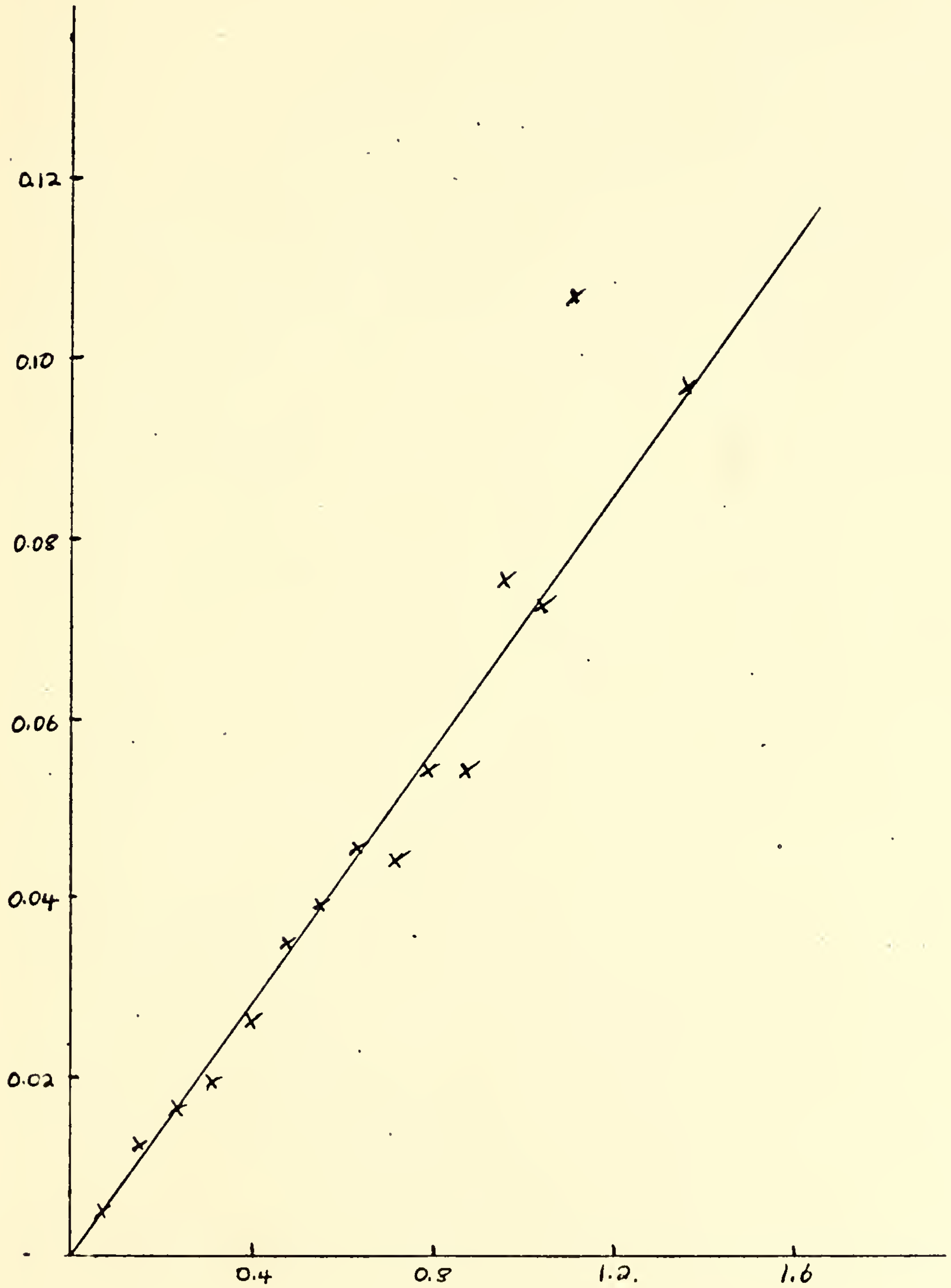


Figure 4. Linear Fit for Variance of Huber Estimator, for q = 0.60.

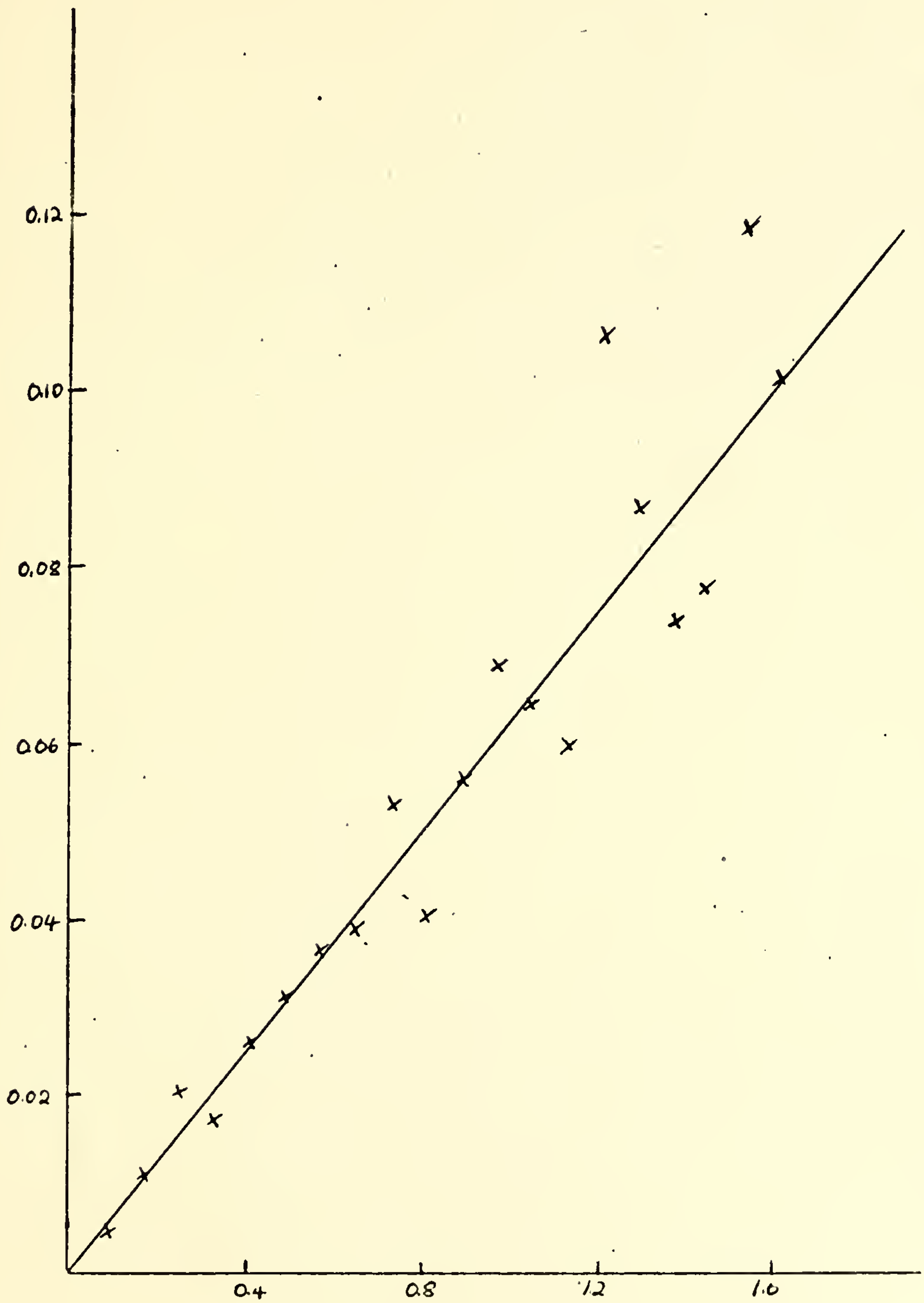


Figure 5. Linear Fit for Variance of Huber Estimator with Transformed Variables, q = 0.9.

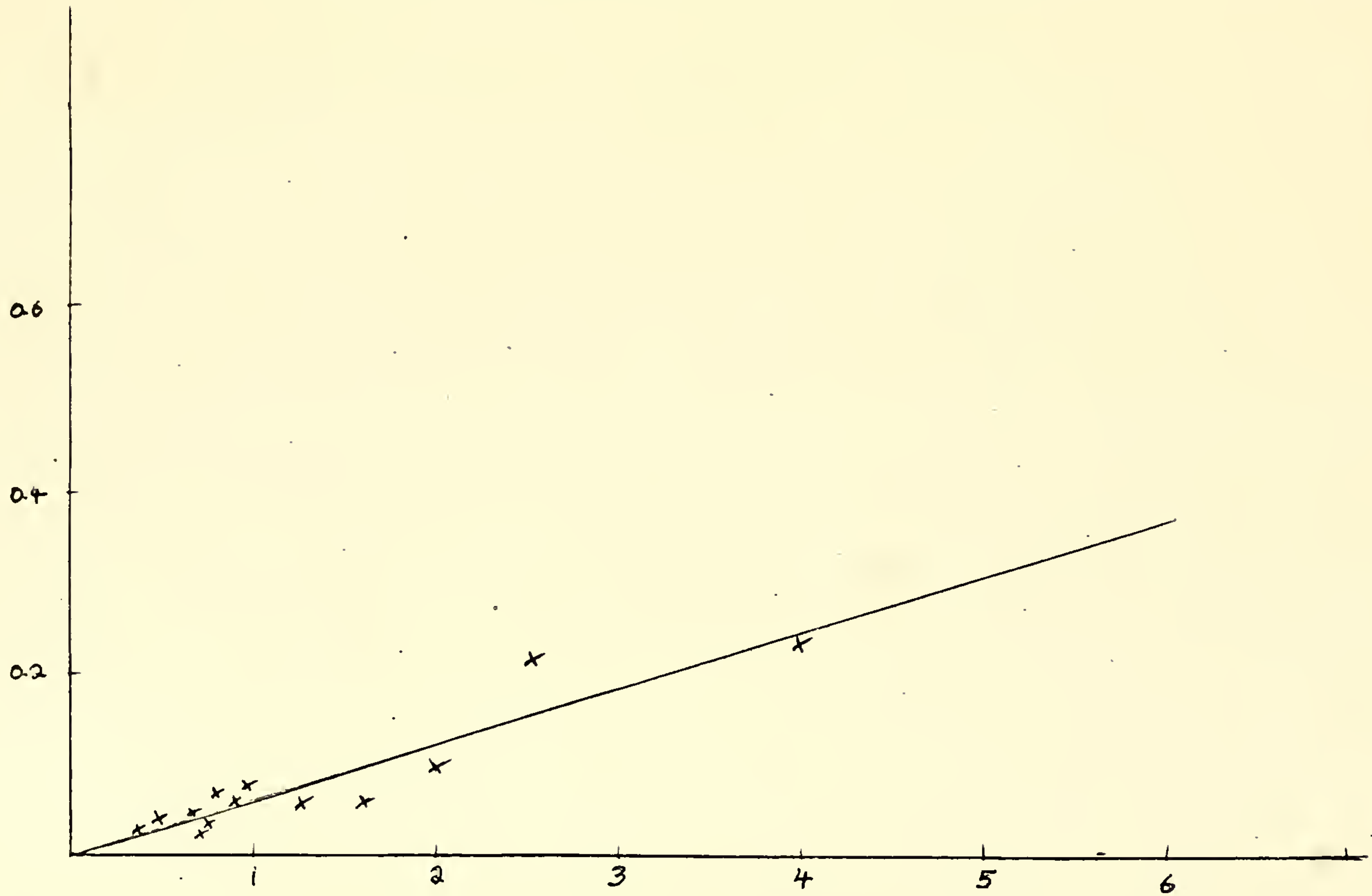


Figure 6. Unweighted Least-Squares, Var(ε) = 0.005.
$\hat{A}$ = 0.572.

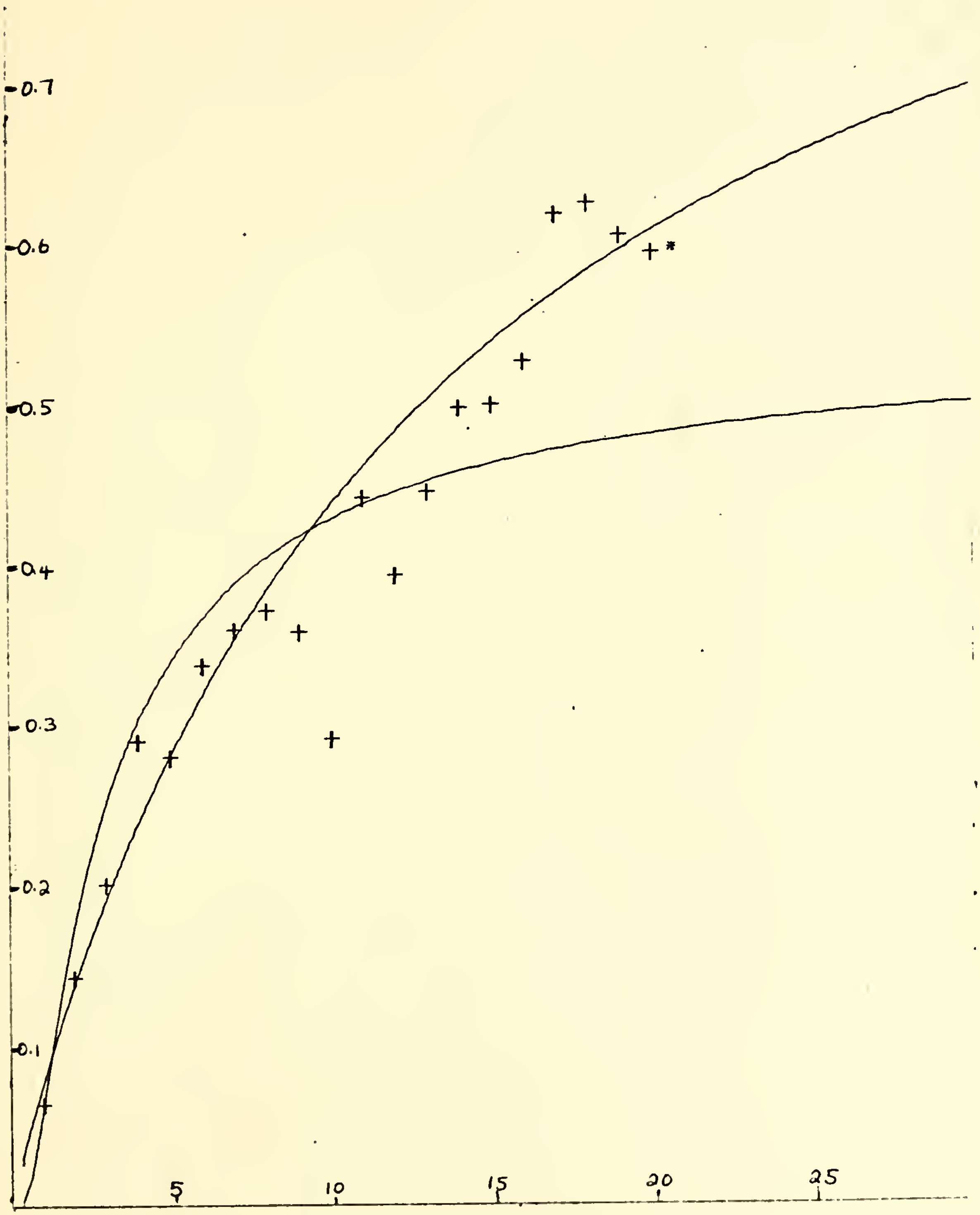


Figure 7. Unweighted Huber, Var(ε) = 0.005, $\hat{A}$ = 0.572.

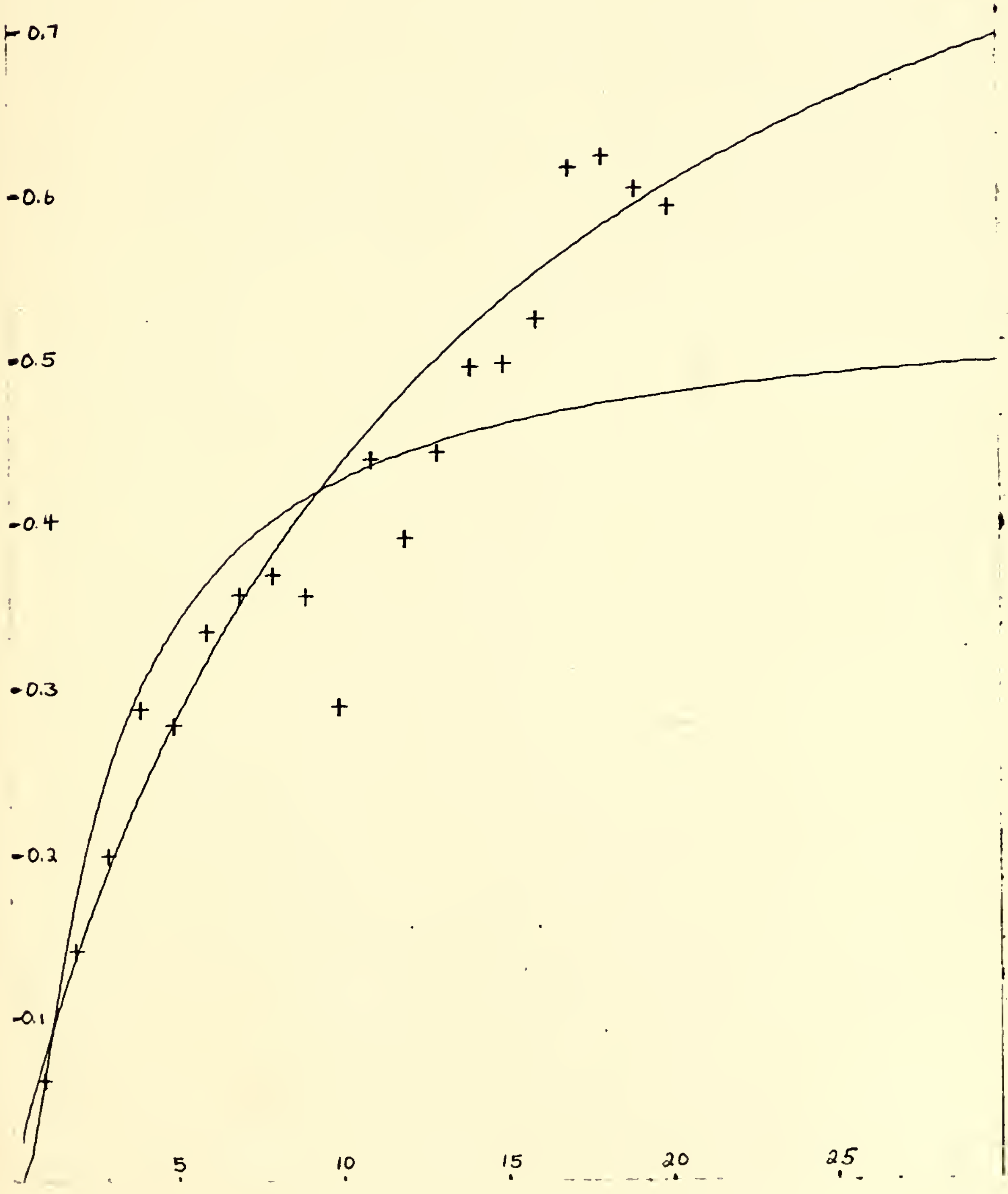


Figure 8. Weighted Least-Squares, var(ε) = 0.005, $w(j)=j^2$
$\hat{A}$ = 0.745.

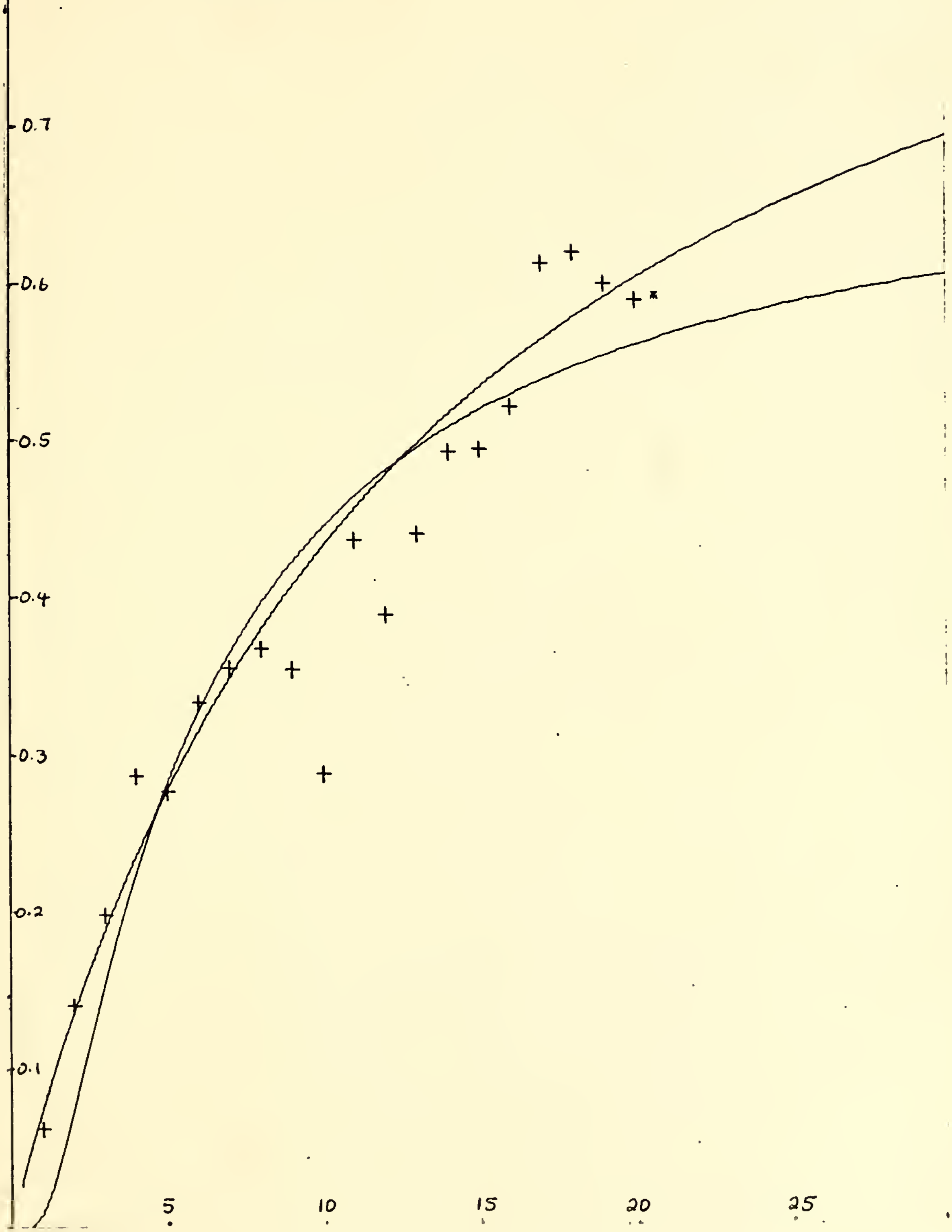


Figure 9. Weighted Huber, var(ε) = 0.005, $w(j) = j^2$,
$\hat{A} = 0.773$.

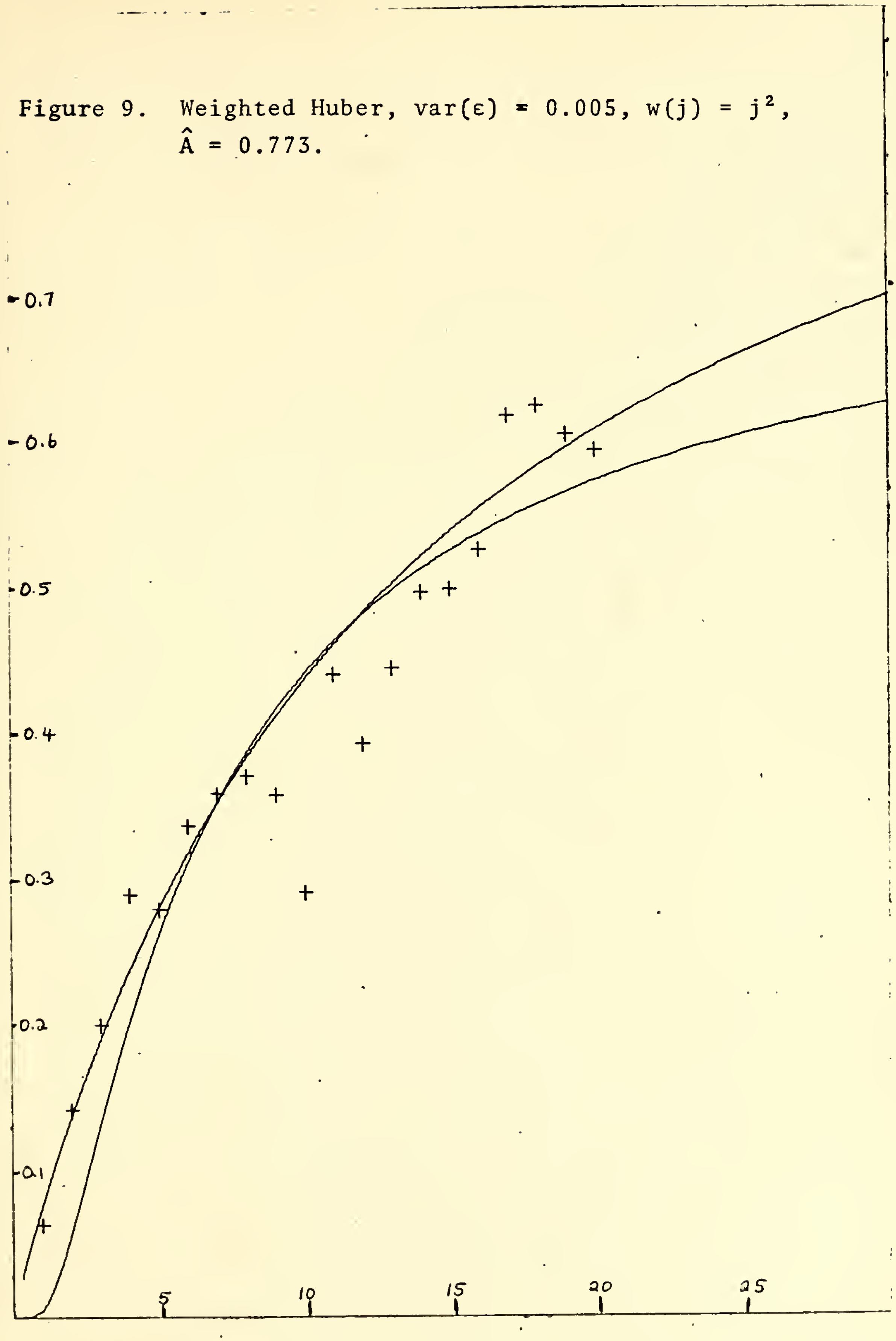


Figure 10. Weighted Least-Squares, var(ε) = 0.005, $w(j) = j^3$

$\hat{A}$ = 0.797.

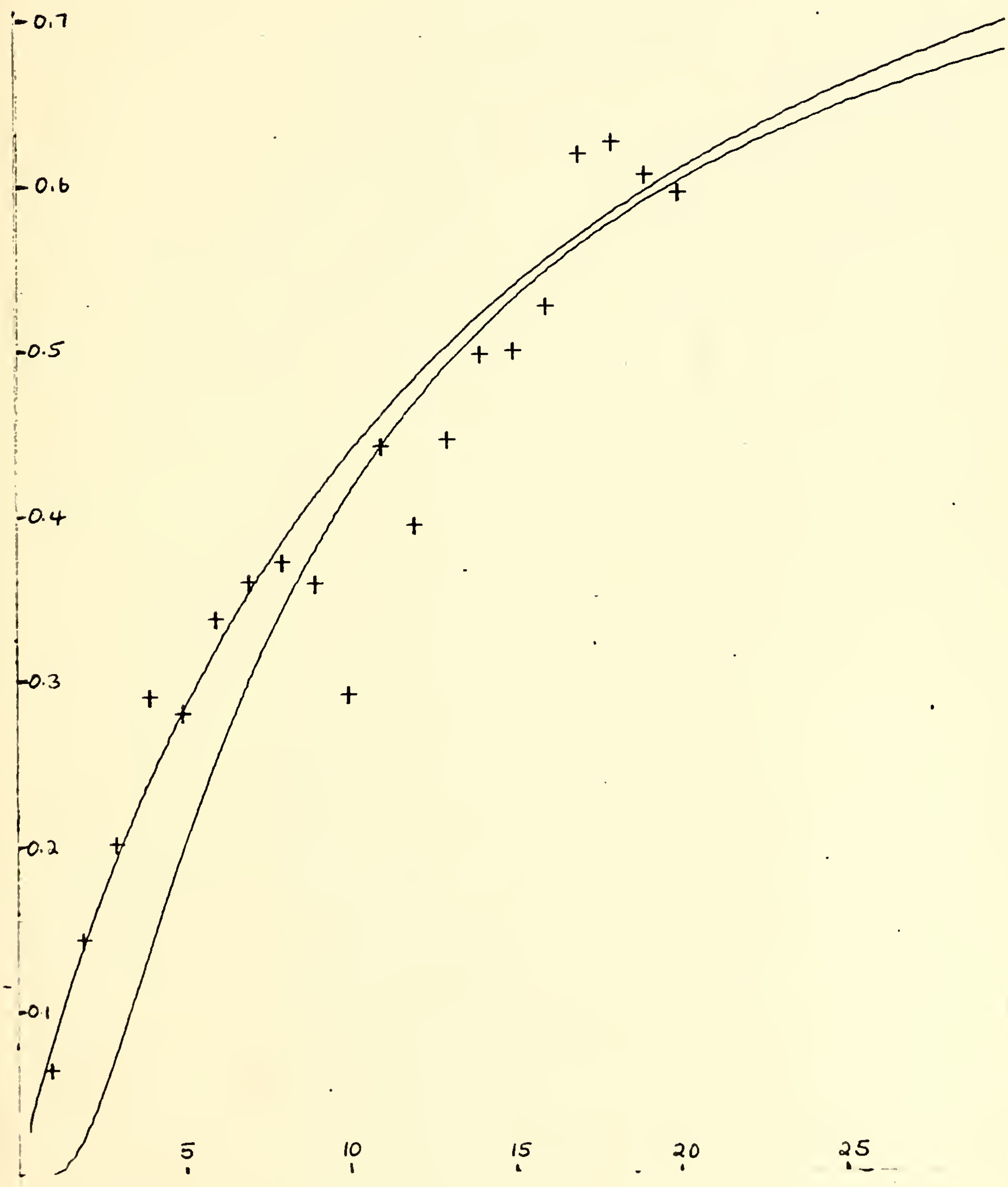


Figure 11. Weighted Huber, $\mathrm{var}(\varepsilon) = 0.005$, $w(j) = j^3$
$\hat{A} = 0.810$.

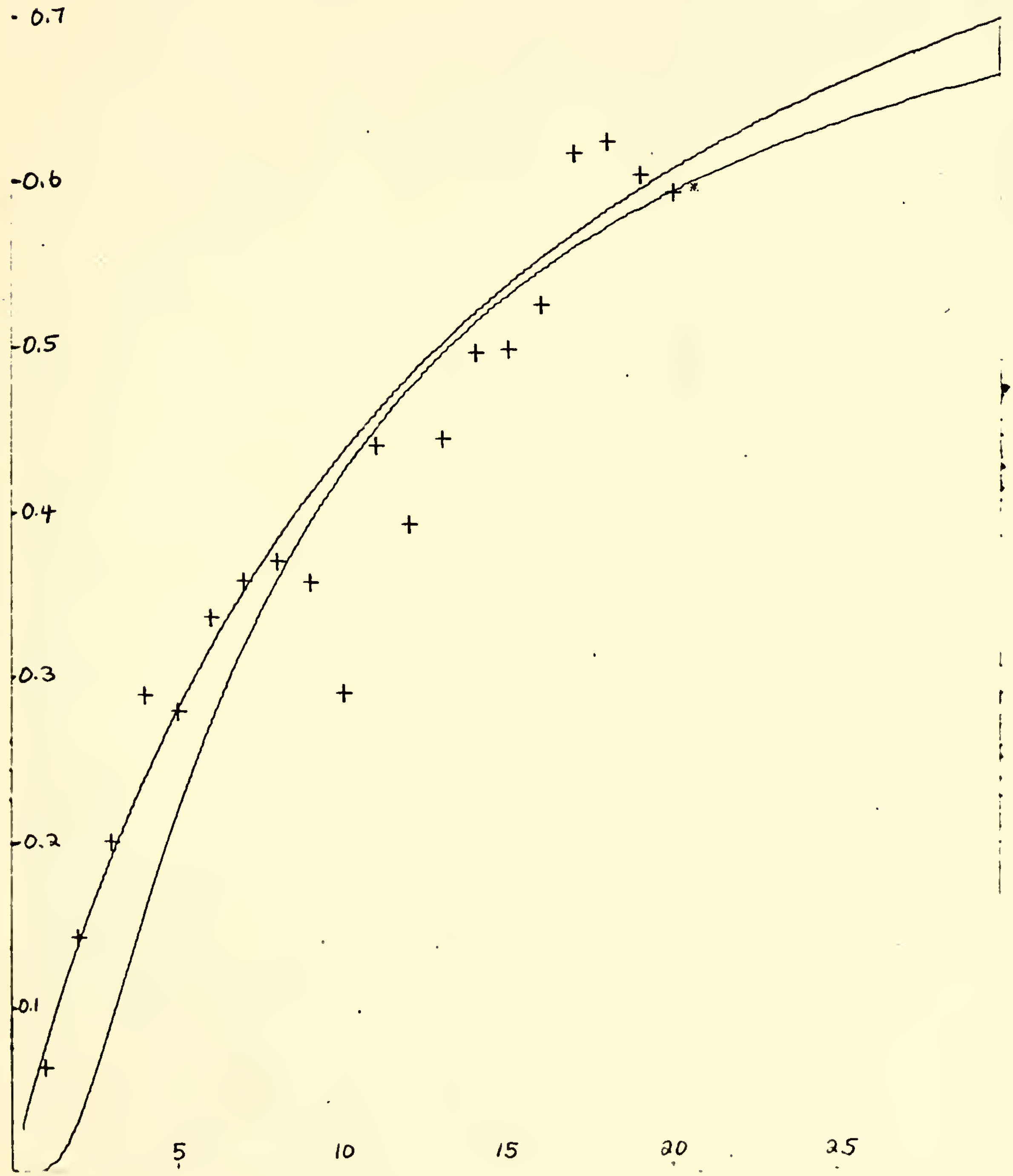


Figure 12. Weighted Least-Squares, $\mathrm{var}(\varepsilon) = 0.005$, $w(j)=j^4$
$\hat{A} = 0.824$.

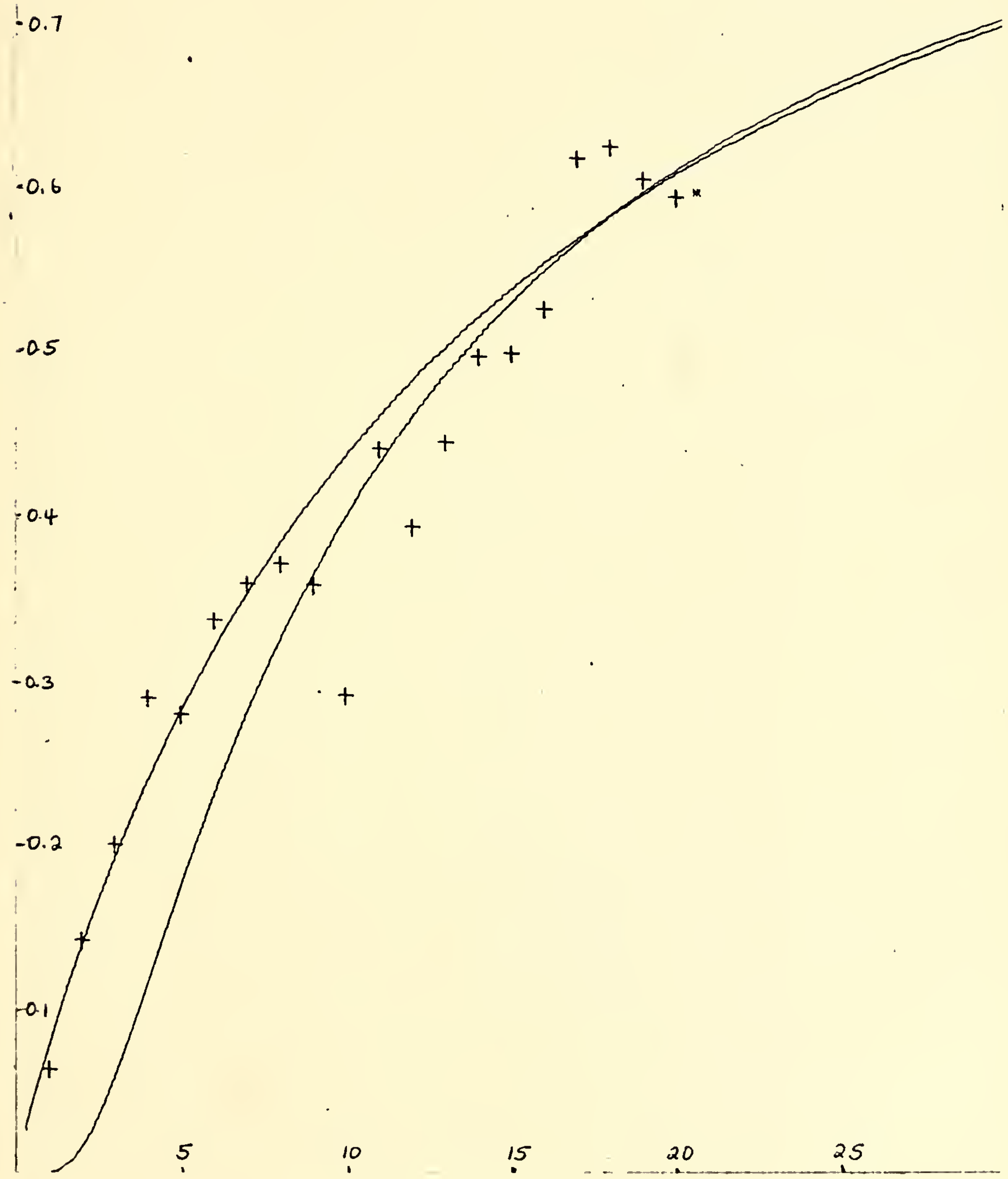


Figure 13. Weighted Huber, $\mathrm{var}(\varepsilon) = 0.005$, $w(j) = j^{*}$
$\hat{A} = 0.828$.

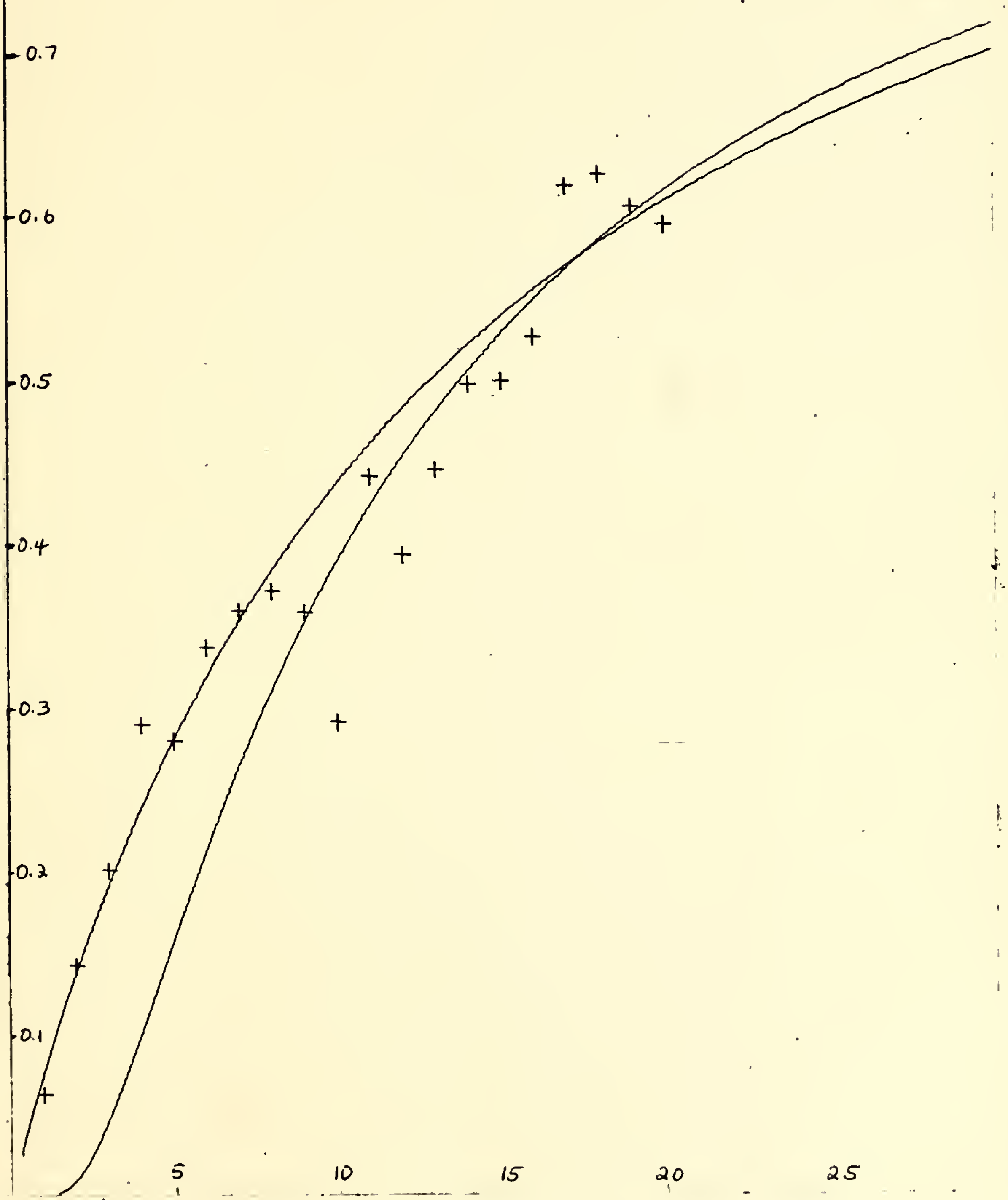


Figure 14. Unweighted Least-Squares, var(ε) = 0.10, $\hat{A}$ = 0.548.

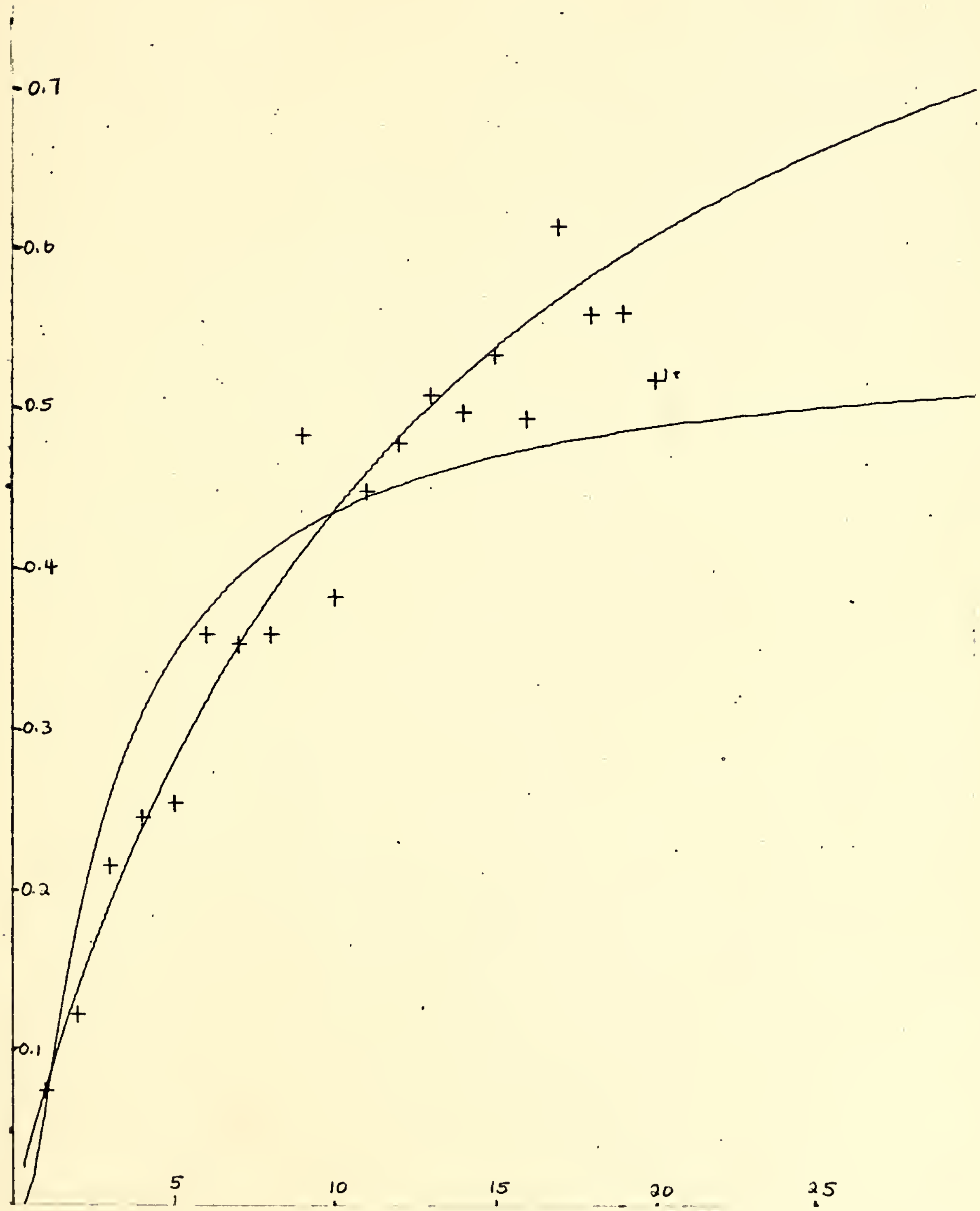


Figure 15. Unweighted Huber, var(ε) = 0.010, $\hat{A}$ = 0.548.

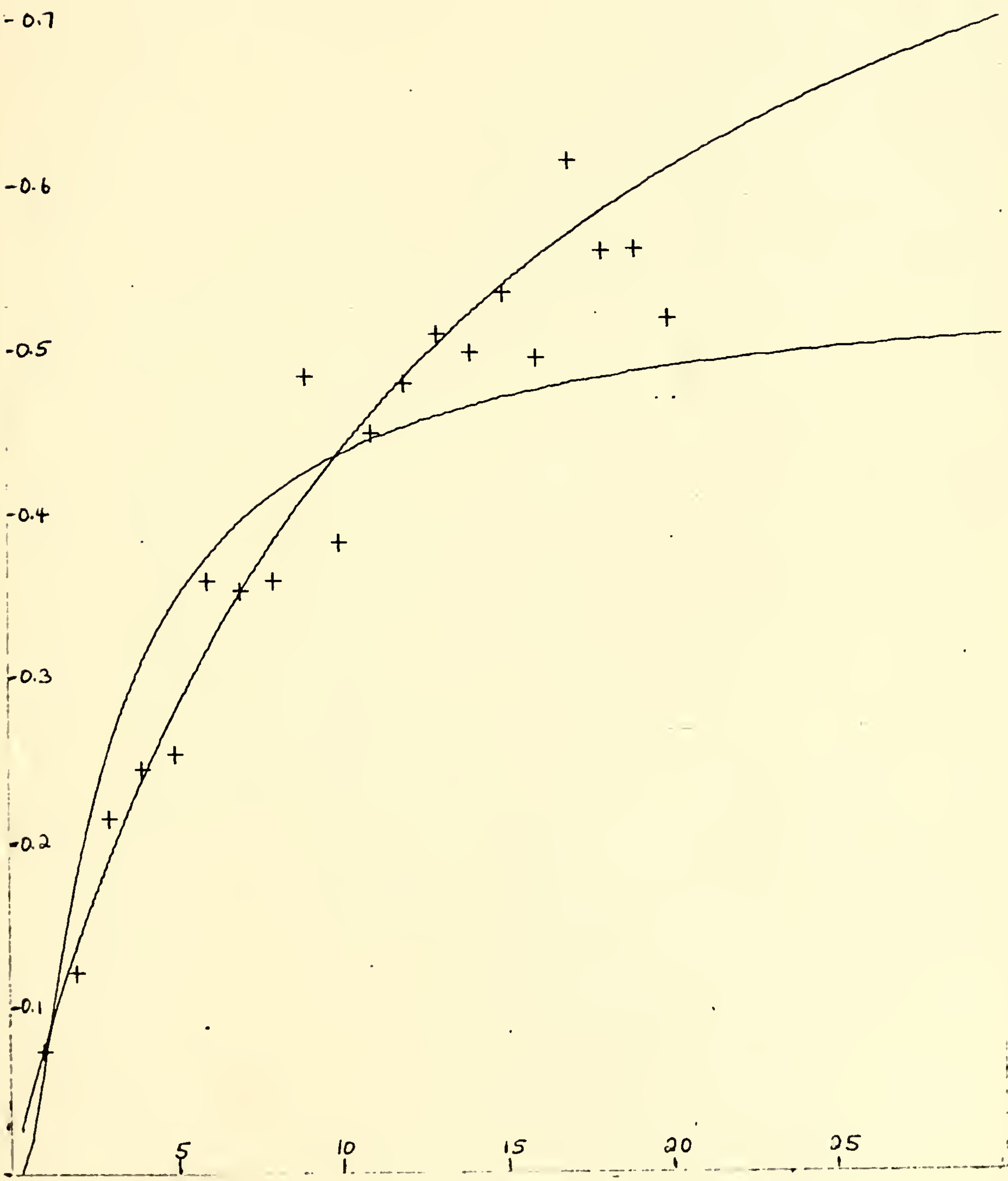


Figure 16. Weighted Least-Squares, var(ε) = 0.010, $w(j) = j^2$, $\hat{A} = 0.664$.

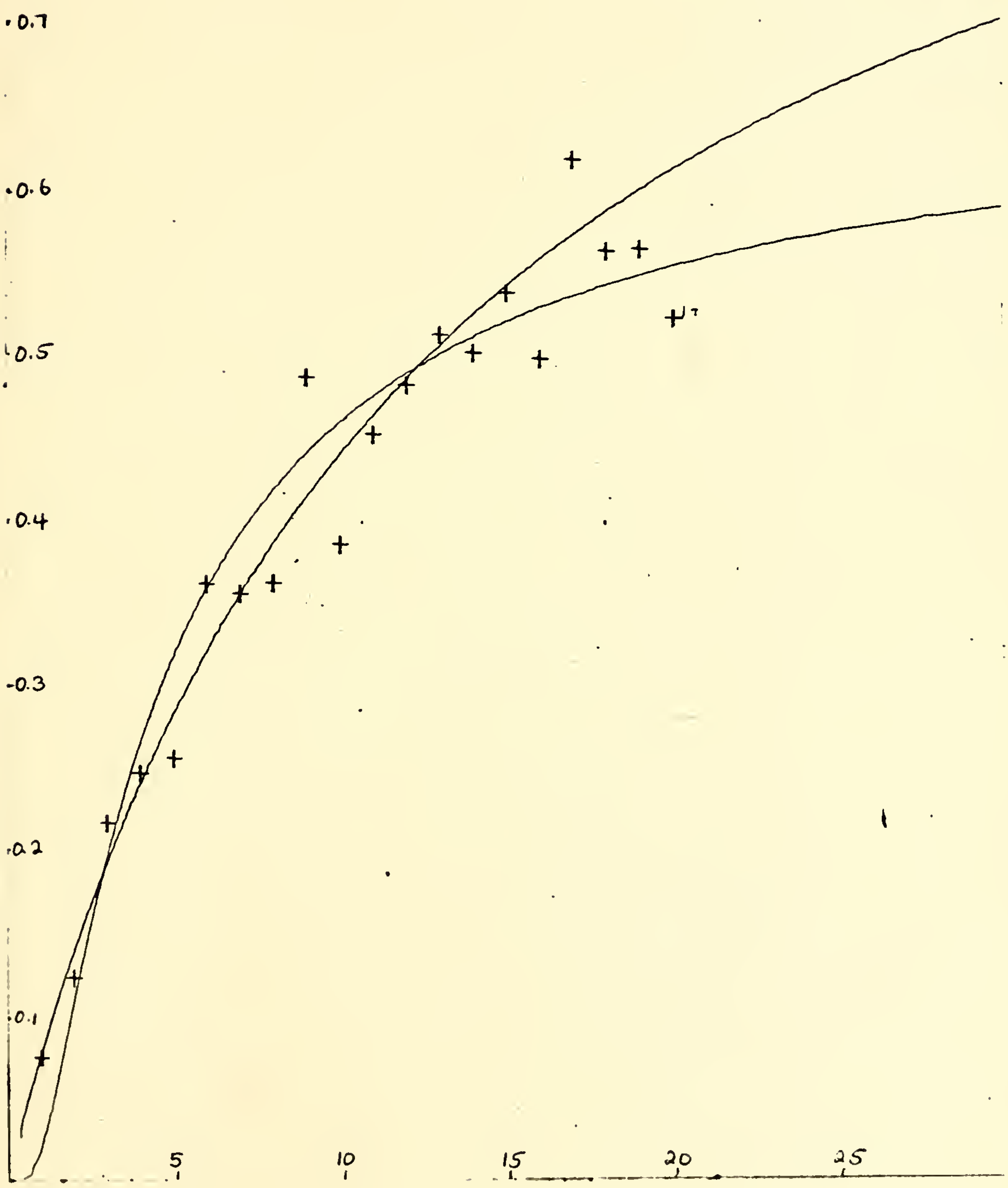


Figure 17. Weighted Huber, $var(\varepsilon) = 0.010$, $w(j) = j^2$
$\hat{A} = 0.678$.

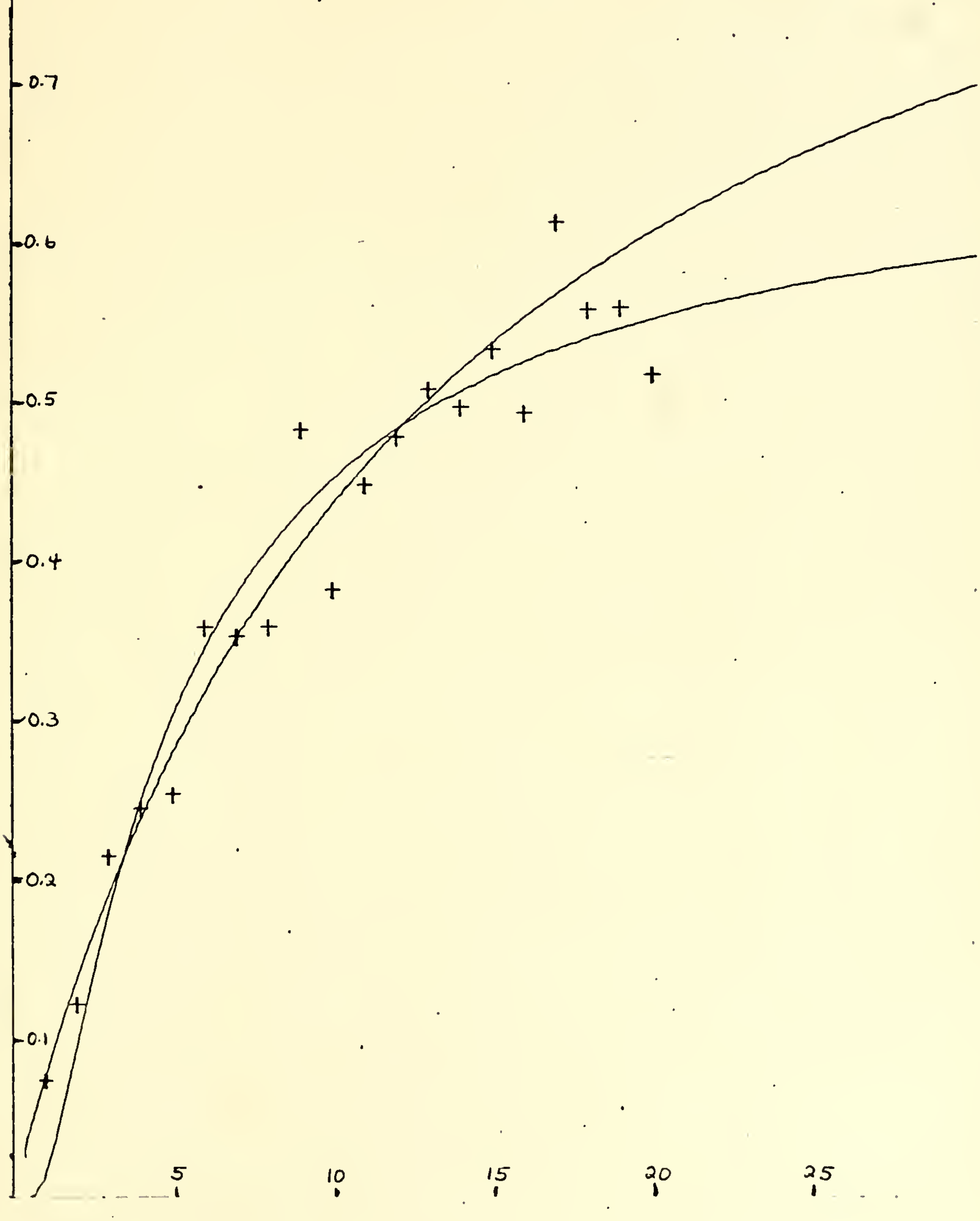


Figure 18. Weighted Least-Squares, var(ε) = 0.010, w(j) = j^2, $\hat{A}$ = 0.695.

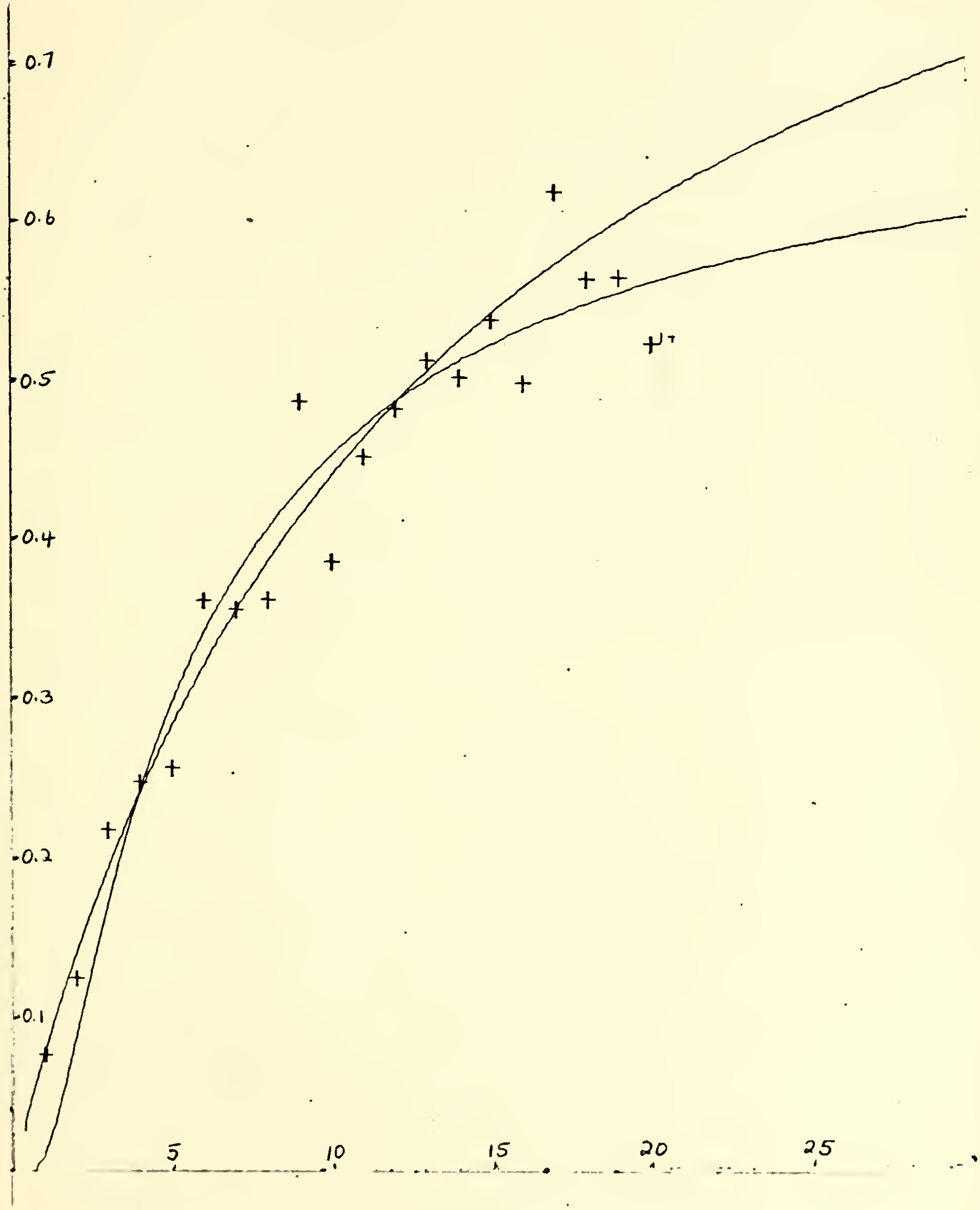


Figure 19. Weighted Huber, $\mathrm{var}(\varepsilon) = 0.01$, $w(j) = j^2$, $\hat{A} = 0.694$.

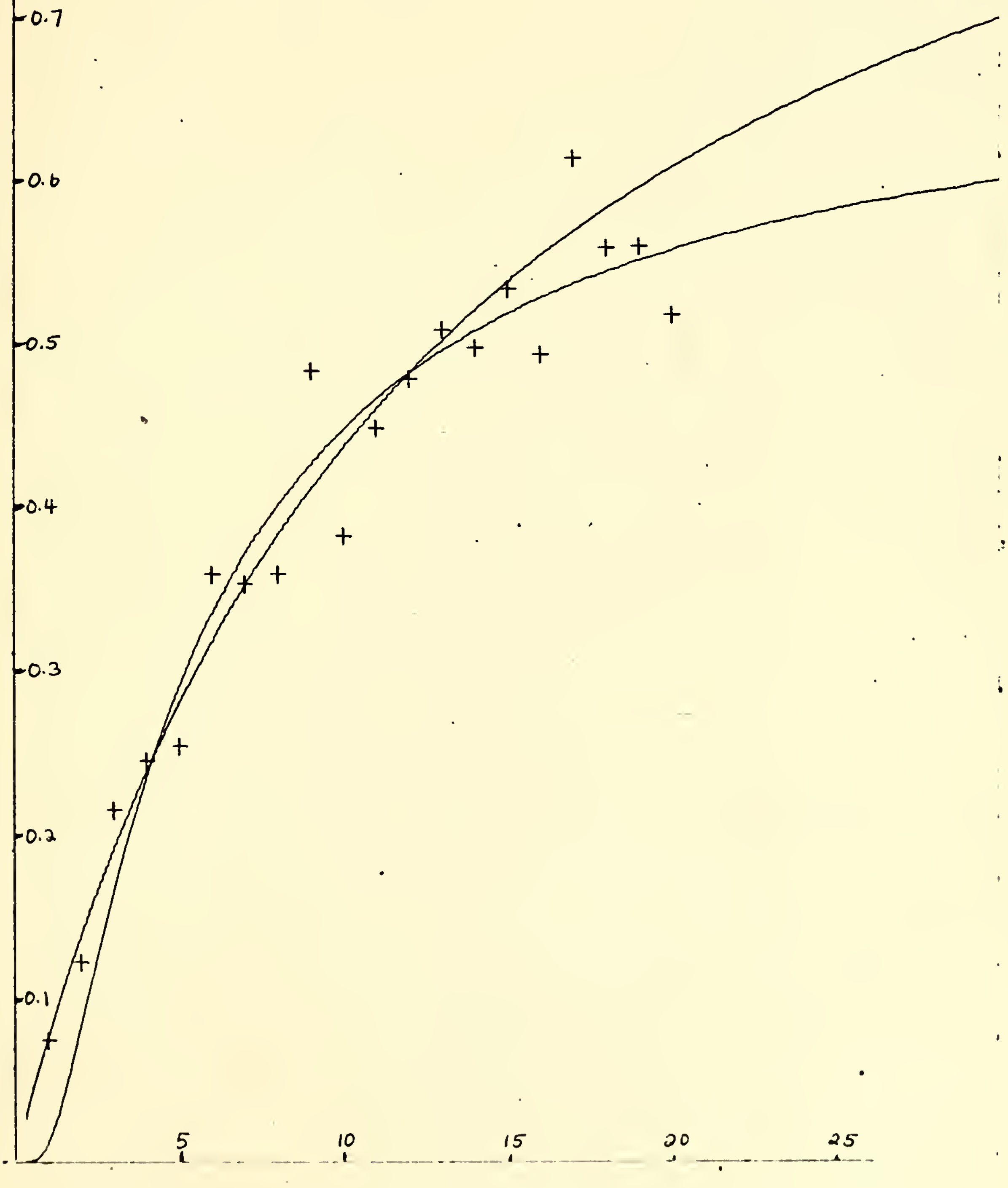


Figure 20. Weighted Least-Squares, var(ε) = 0.010, w(j) = j^4, $\hat{A}$ = 0.691.

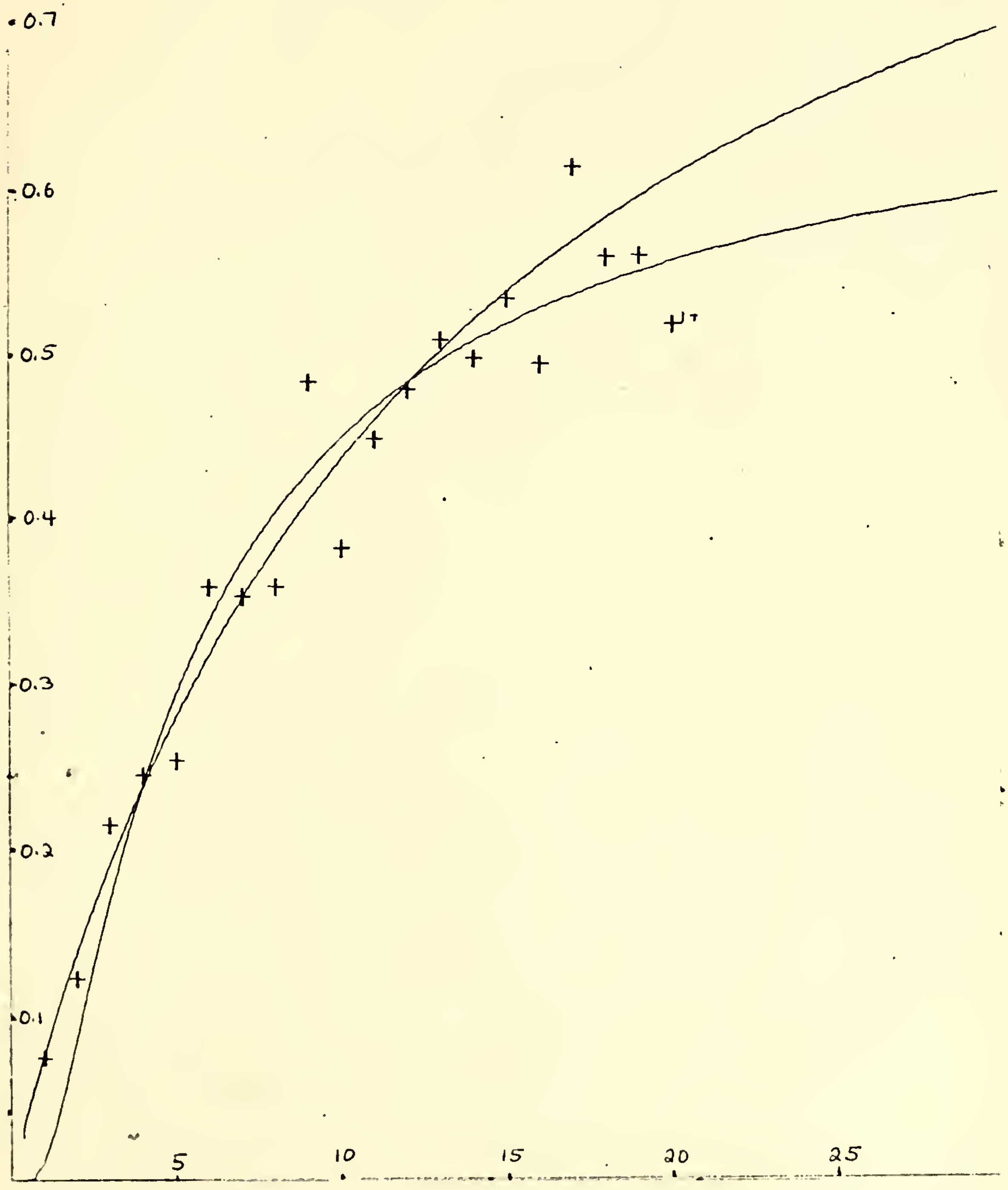
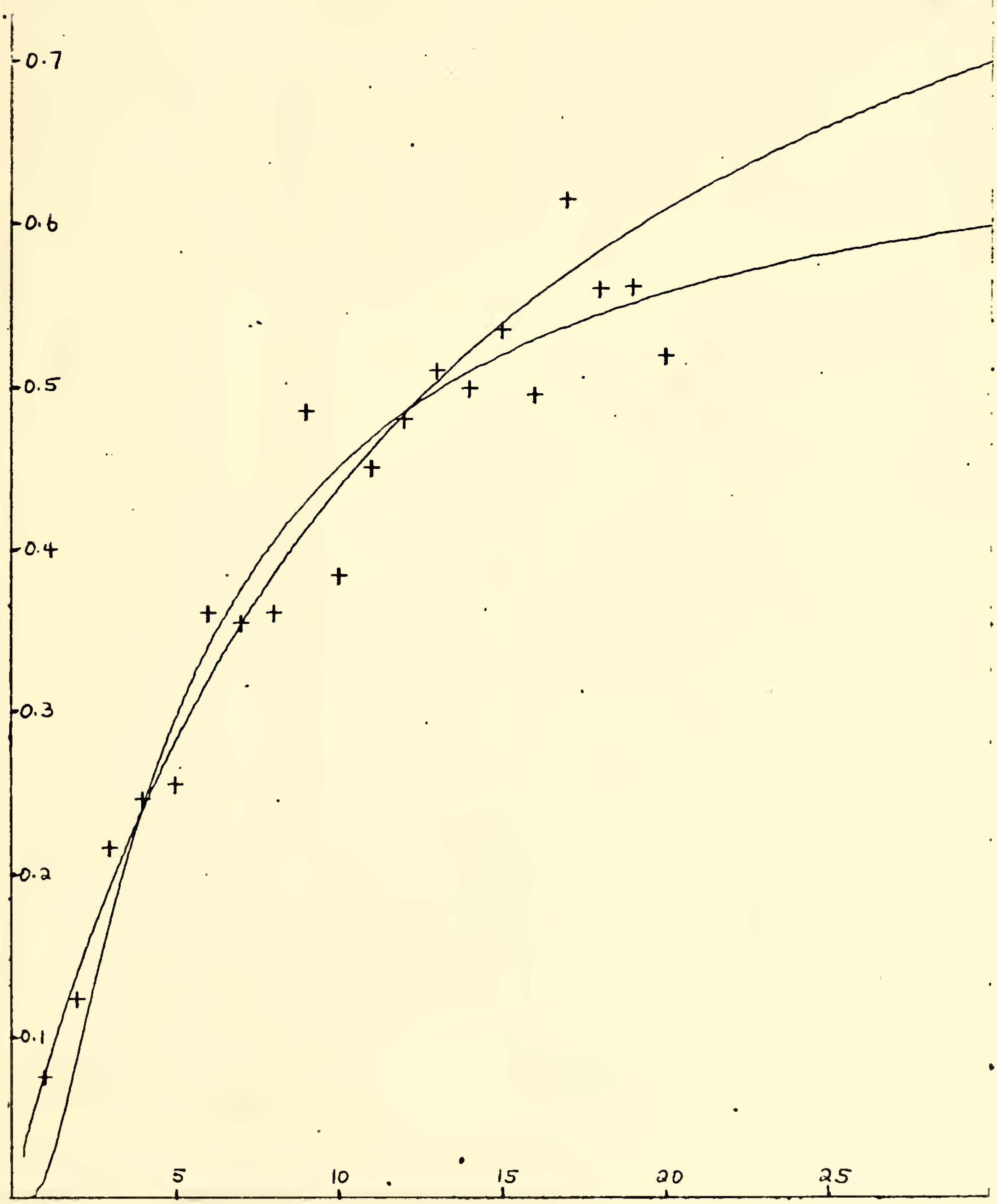


Figure 21. Weighted Huber, var(ε) = 0.010, $w(j) = j^4$
$\hat{A} = 0.691$.



```
C     .........................................................................
C
C
C     1         WEIGHTED LEAST-SQUARES FOR THE MODEL  Y=A*EXP(-B/T)*EXP(U)
C
C     DATA             Y = FTRUE*(CTRUE*T/(1.0+CTRUE*T)
C     IX           RANDOM NUMBER SEED
C     M            RANGE OF T IS FROM 1 TO M
C     LL           FOUR DATA CARDS WITH ONE VALUE OF RHO ON EACH CARD
C     RHO(LL)      IS LINEAR FUNCTION OF SCALE PARAMETER
C     PROB         THIS NUMBER IS USED TO TO DETERMINE THE PERCENTAGE OF
C                  TWO DENSITIES
C     WSHOT(I)     THE ASSUMED DISTRIBUTION
C
      DIMENSION X(100),Y(100),YY(100),KEY(100),T(20),INDEX(20),
     1          RHO(20),W(20),ZZ(40),WSHOT(100),Z(20),CAUCHY(20),
     2          ALG(100),A(100),U(20),P(100),AREA(100)
      ALP = 0.85
      IX = 85431
      N = 100
      M = 20
      PROB = 0.9
      FTRUE = 1.0
      CTRUE = 0.0772
      BTRUE = 10.0
      CALL ERRSET(208,256,0,1,1,209)
      CALL OVFLOW
      DO 1111 I = 1,M
      W(I) = ALP**(20-I)
 1111 CONTINUE
      READ(5,1777) (RHO(I),I=1,4)
 1777 FORMAT(F10.4)
      DO 8000 LL = 1,4
      WRITE(6,1666) RHO(LL)
 1666 FORMAT('1',9X,'RHO= ',F8.4)
C
      DO 9000 K = 1,N
      CALL SNORM(IX,Z,M)
      CALL RANDOM(IX,U,M)
      DO 3000 I = 1,M
      IF (U(I).LT.PROB) GO TO 3001
      WSHOT(I) = RHO(LL) * 3.0 * Z(I)
      GO TO 3000
 3001 WSHOT(I) = RHO(LL) * Z(I)
 3000 CONTINUE
      DO 3100 I = 1,M
      T(I) = I
      YY(I) = FTRUE * ((CTRUE*T(I))/(1.0+CTRUE*T(I)))*EXP(WSHOT(I))
```

```
      Y(I) = ALOG(YY(I))
      X(I) = -(1.0/T(I))
 3100 CONTINUE
C
      SUMW = 0.0
      SUMWY = 0.0
      SUMWX = 0.0
      SUMWXY = 0.0
      SUMWX2 = 0.0
      DO 1000 I = 1,M
      SUMW = SUMW + W(I)
      SUMWY = SUMWY + W(I)*Y(I)
      SUMWXY = SUMWXY + W(I)*X(I)*Y(I)
      SUMWX = SUMWX + W(I)*X(I)
      SUMWX2 = SUMWX2 + W(I)*X(I)**2
 1000 CONTINUE
      C = SUMWY*SUMWX2 - SUMWXY*SUMWX
      D = SUMW*SUMWX2 - SUMWX**2
      ALG(K) = C/D
      A(K) = EXP(ALG(K))
 9000 CONTINUE
      CALL HISTG(ALG,N,0)
      CALL HISTG(A,N,0)
      CALL SHSORT(ALG,KEY,N)
      CALL SHSORT(A,KEY,N)
      WRITE(6,1999) ((I,ALG(I),A(I)),I=1,N)
 1999 FORMAT('0',I8,2F12.8)
 8000 CONTINUE
      STOP
      END
```

```
C   ...................................................................
C
C
C   2       WEIGHTED LEAST-SQUARES FOR THE MODEL Y=A*EXP(-B/T**C)*EXP(U)
C
C   DATA             Y = FTRUE*(CTRUE*T/(1.0+CTRUE*T)
C   IX          RANDOM NUMBER SEED
C   IT          NUMBER OF ITERATIONS
C   M           RANGE OF T IS FROM 1 TO M
C   RHO(LL)     IS LINEAR FUNCTION OF SCALE PARAMETER
C   PROB        THIS NUMBER IS USED TO TO DETERMINE THE PERCENTAGE OF
C               TWO DENSITIES
C   WSHOT(I)    THE ASSUMED DISTRIBUTION
C
      DIMENSION T(20),X(20),Y(20),YY(20),RHO(30),U(20),WSHOT(20),A(100),
     1   B(100),C(100),A1(100),B1(100),KEY(100),Z(40)
      IX = 531278
      N = 100
      IT = 15
      M = 20
      FTRUE = 1.0
      BTRUE = 10.0
      CTRUE = 0.0772
      PROB = 0.9
      CALL ERRSET(208,256,0,1,1,209)
      CALL OVFLOW
      READ(5,1777) (RHO(I),I=1,3)
 1777 FORMAT(F10.4)
      DO 8000 LL = 1,3
      WRITE(6,1666) RHO(LL)
 1666 FORMAT('1',9X,'RHO= ',F8.4)
C
      DO 9000 K = 1,N
      CALL SNORM(IX,Z,M)
      CALL RANDOM(IX,U,M)
      DO 3000 I = 1,M
      IF (U(I).LT.PROB) GO TO 3001
      WSHOT(I) = RHO(LL)*3.0*Z(I)
      GO TO 3000
 3001 WSHOT(I) = RHO(LL) * Z(I)
 3000 CONTINUE
C
      DO 3100 I = 1,M
      T(I) = I
      YY(I) = FTRUE * ((CTRUE*T(I))/(1.0+CTRUE*T(I)))*EXP(WSHOT(I))
      Y(I) = ALOG(YY(I))
      X(I) = -(1.0/T(I))
 3100 CONTINUE
```

```
C
      CALL REG(X,Y,ALPHA,BETA,SUMY,VAR,M,1)
      CALL ITRATE(Y,SUMY,AO,BO,CO,M,IT,ALPHA,BETA)
      A(K) = AO
      B(K) = BO
      C(K) = CO
      A1(K) = EXP(ALPHA)
      B1(K) = BETA
 9000 CONTINUE
      WRITE(6,5000) RHO(LL)
 5000 FORMAT('1',F10.4)
      CALL HISTG(A,N,0)
      WRITE(6,4100)
 4100 FORMAT('0','HISTOGRAM OF A')
      CALL HISTG(B,N,0)
      WRITE(6,4200)
 4200 FORMAT('0','HISTOGRAM OF B')
      CALL HISTG(C,N,0)
      WRITE(6,4300)
 4300 FORMAT('0','HISTOGRAM OF C')
      CALL HISTG(A1,N,0)
      WRITE(6,4400)
 4400 FORMAT('0','HISTOGRAM OF A1')
      CALL HISTG(B1,N,0)
      WRITE(6,4500)
 4500 FORMAT('0','HISTOGRAM OF B1')
      CALL SHSORT(A,KEY,N)
      CALL SHSORT(B,KEY,N)
      CALL SHSORT(C,KEY,N)
      CALL SHSORT(A1,KEY,N)
      CALL SHSORT(B1,KEY,N)
      WRITE(6,5100)
 5100 FORMAT('0',7X,'A',9X,'B',9X,'C',8X,'A1',8X,'B1')
      DO 5300 I = 1,N
      WRITE(6,5200) A(I),B(I),C(I),A1(I),B1(I)
 5200 FORMAT(5(F10.4))
 5300 CONTINUE
 8000 CONTINUE
      STOP
      END
C
      SUBROUTINE ITRATE(Y,SUMY,A,B,C,M,IT,ALPHA,BETA)
      DIMENSION Y(1),AA(20),BB(20),CC(20),ALOGJ(20),STORE(20),W(20)
      ALP = 0.7
      AA(1) = 0.223
      BB(1) = 3.5
      CC(1) = 0.5
      SUM = 0.0
```

```
      SUM9 = 0.0
      DO 500 J = 1,M
      ALOGJ(J) = ALOG(FLOAT(J))
      W(J) = ALP**(20-J)
      SUM = SUM + W(J)
      SUM9 = SUM9 + W(J)*Y(J)
  500 CONTINUE
      DO 9000 I = 1,IT
      SUM1 = 0.0
      SUM2 = 0.0
      SUM3 = 0.0
      SUM4 = 0.0
      SUM5 = 0.0
      SUM6 = 0.0
      SUM7 = 0.0
      SUM8 = 0.0
      DO 1000 J = 1,M
      SUM1 = SUM1 + W(J)/FLOAT(J)**CC(I)
      SUM2 = SUM2 + (W(J)*ALOGJ(J))/FLOAT(J)**CC(I)
 1000 CONTINUE
      CC(I+1) = CC(I)- (SUM9-SUM*AA(I)+BB(I)*SUM1)/(-BB(I)*SUM2)
      DO 2000 J = 1,M
      SUM3 = SUM3 + (W(J)*Y(J))/FLOAT(J)**CC(I+1)
      SUM4 = SUM4 + W(J)/FLOAT(J)**CC(I+1)
      SUM5 = SUM5 + W(J)/FLOAT(J)**(2*CC(I+1))
 2000 CONTINUE
      BB(I+1)=BB(I)  - (SUM3-AA(I)*SUM4+BB(I)*SUM5)/SUM5
      DO 3000 J = 1,M
      SUM6 = SUM6 + (W(J)*Y(J)*ALOGJ(J))/FLOAT(J)**CC(I+1)
      SUM7 = SUM7 + W(J)*Y(J)*ALOGJ(J)
      SUM8 = SUM8 + W(J)/(J*FLOAT(J)**(2*CC(I+1)))
 3000 CONTINUE
      AA(I+1) = AA(I) - (SUM6-AA(I)*SUM7+BB(I+1)*SUM8)/(-SUM7)
 9000 CONTINUE
      A = EXP(AA(IT+1))
      B = BB(IT+1)
      C = CC(IT+1)
      RETURN
      END
      SUBROUTINE REG(X,Y,ALPHA,BETA,SUMY,VAR,M,NFLAG)
      DIMENSION X(1),Y(1)
      SUMSQX = 0.0
      SUMXY = 0.0
      SUMX = 0.0
      SUMY = 0.0
      DO 1000 J = 1,M
      SUMX = SUMX + X(J)
      SUMY = SUMY + Y(J)
```

```
      SUMXY = SUMXY + X(J)*Y(J)
      SUMSQX = SUMSQX + X(J)**2
1000  CONTINUE
      XBAR = SUMX/FLOAT(M)
      YBAR = SUMY/FLOAT(M)
      A = SUMXY - M*XBAR*YBAR
      B = SUMSQX - M*XBAR**2
      BETA = A/B
      ALPHA = YBAR - BETA*XBAR
      IF (NFLAG.EQ.1) GO TO 3000
      VAR = 0.0
      DO 2000 J = 1,M
      VAR = VAR + (Y(J) - ALPHA - BETA*X(J))**2
2000  CONTINUE
      VAR = VAR/FLOAT(M-2)
3000  RETURN
      END
```

```
C ...........................................................................
C
C
C    3        WEIGHTED HUBER METHOD FOR THE MODEL Y=A*EXP(-B/T)*EXP(U)
C
C    DATA
       YY(I)  =  FTRUE * ((CTRUE*T(I))/(1.0+CTRUE*T(I)))*EXP(WSHOT(I))
C    WSHOT(I)   THE ASSUMED DISTRIBUTION
C               TWO DENSITIES
C    PROB       THIS NUMBER IS USED TO TO DETERMINE THE PERCENTAGE OF
C    RHO(LL)    IS LINEAR FUNCTION OF SCALE PARAMETER
C    LL         FOUR DATA CARDS WITH ONE VALUE OF RHO ON EACH CARD
C    M          RANGE OF T IS FROM 1 TO M
C    IX         RANDOM NUMBER SEED
C    EXPD(K)    WEIGHTED HUBER ESTIMATE OF A
C    EXPDD(K)   UNWEIGHTED LEAST SQUARES ESTIMATE OF A
C    N          NUMBER OF SAMPLES OF A
C
C
      DIMENSION X(100),Y(100),A(100,20),B(100,20),D(100,20),
     1   INDEX(100),AAA(100),BBB(100),SSS(100),KEY(100),F(200),
     2   R(100),W(100),ZZ(200),WSHOT(100),Z(600),CAUCHY(100),
     3   S(100,20),DD(100),DDD(100),EXPD(100),YY(100),T(100),U(100),
     4   P(100),AREA(100),KAY(100),EXPDD(100),RHO(20)
      ALP = 0.7
      IX = 38419
      N = 100
      M = 20
      FTRUE = 1.0
      CTRUE = 0.0772
      PROB = 0.9
      CALL ERRSET (208,256,0,1,1,209)
      WATE = 0.0
      DO 100 I = 1,20
      WATE = WATE + I**4
  100 CONTINUE
      READ(5,1777) (RHO(I),I=1,3)
 1777 FORMAT(F10.4)
      DO 9100 I = 1,N
      INDEX(I) = I
 9100 CONTINUE
      CALL OVFLOW
      DO 8000 LL = 1,3
      WRITE(6,1666) RHO(LL)
 1666 FORMAT('1',9X,'RHO = ',F8.4)
      DO 9000 K = 1,N
      CALL SNORM(IX,Z,M)
      CALL RANDOM(IX,U,M)
```

```
      DO 3000 I = 1,N
      IF (U(I).LT.PROB) GO TO 3001
      WSHOT(I) = RHO(LL) * 3.0 * Z(I)
      GO TO 3000
3001  WSHOT(I) = RHO(LL) * Z(I)
3000  CONTINUE
      DO 3100 I = 1,M
      T(I) = I
      YY(I) = FTRUE * ((CTRUE*T(I))/(1.0+CTRUE*T(I)))*EXP(WSHOT(I))
      Y(I) = ALOG(YY(I))
      X(I) = -(1.0/T(I))
3100  CONTINUE
      SUMX = 0.0
      SUMY = 0.0
      DO 1000 I = 1,M
      SUMX = SUMX + X(I)
      SUMY = SUMY + Y(I)
1000  CONTINUE
      XBAR = SUMX/FLOAT(M)
      YBAR = SUMY/FLOAT(M)
C     COMPUTES THE VALUES OF A,B,S,S**2, AND D
      SSN2 = 0.0
      SUMXY = 0.0
      SUMSQX = 0.0
      DO 1100 I = 1,M
      SUMXY = SUMXY + X(I)*Y(I)
      SUMSQX = SUMSQX + X(I)**2
1100  CONTINUE
      AO = SUMXY - M*XBAR*YBAR
      BO = SUMSQX - FLOAT(M)*XBAR**2
      A(K,1) = YBAR
      B(K,1) = AO/BO
      DO 1500 I = 1,M
      SSN2 = SSN2 + (Y(I)-A(K,1)-B(K,1)*(X(I)-XBAR))**2
1500  CONTINUE
      S(K,1) = SQRT(SSN2/FLOAT(M-2))
      D(K,1) = A(K,1) - B(K,1)*XBAR
      DO 4000 J = 1,19
      SUM11 = 0.0
      SUM21 = 0.0
      SUM31 = 0.0
      SUM41 = 0.0
      SUM51 = 0.0
      SUM61 = 0.0
      DO 4100 I = 1,M
      R(I) = Y(I) - A(K,J) - B(K,J)*(X(I)-XBAR)
      R(I) = R(I)/S(K,J)
      IF (R(I).LE.2.0.AND.R(I).GE.-2.0) GO TO 4200
```

```
      IF (R(I).GT.2.0) GO TO 4300
      IF (R(I).LT.-2.0) F(I) = -2.0
      GO TO 4400
 4200 F(I) = R(I)
      GO TO 4400
 4300 F(I) = 2.0
 4400 W(I) = ((I**4*F(I))/WATE)/
     1        (R(I)/S(K,J))
      SUM11 = SUM11 +W(I)
      SUM21 = SUM21 + W(I)*(X(I)-XBAR)
      SUM31 = SUM31 + W(I)*Y(I)
      SUM41 = SUM41 + W(I)*(X(I)-XBAR)**2
      SUM51 = SUM51 + W(I)*Y(I)*(X(I)-XBAR)
      SUM61 = SUM61 + W(I)*R(I)**2
 4100 CONTINUE
      L = J+1
      A(K,L) = (SUM31*SUM41-SUM21*SUM51)/(SUM11*SUM41-SUM21**2)
      B(K,L) = (SUM11*SUM51-SUM31*SUM21)/(SUM11*SUM41-SUM21**2)
      S(K,L) = SQRT(SUM61)
      D(K,L) = A(K,L) - B(K,L)*XBAR
 4000 CONTINUE
      AAA(K) = A(K,20)
      BBB(K) = B(K,20)
      DDD(K) = D(K,20)
      SSS(K) = S(K,20)
      EXPD(K) = EXP(DDD(K))
      DD(K) = D(K,1)
      EXPDD(K) = EXP(DD(K))
      WRITE(6,6999) AAA(K),BBB(K),SSS(K),DDD(K),EXPD(K),DD(K),EXPDD(K)
 6999 FORMAT(' ',9X,7F10.6)
 9000 CONTINUE
      CALL HISTG(DDD,N,0)
      WRITE(6,4009)
 4009 FORMAT('0','HISTOGRAM OF DDD')
      CALL HISTG(EXPD,N,0)
      WRITE(6,5009)
 5009 FORMAT('0','HISTOGRAM OF EXPD')
      CALL HISTG(DD,N,0)
      WRITE(6,6009)
 6009 FORMAT('0','HISTOGRAM OF DD')
      CALL HISTG(EXPDD,N,0)
      WRITE(6,7009)
 7009 FORMAT('0','HISTOGRAM OF EXPDD')
C
C     SORTS AAA,BBB,SSS,DDD,EXPD AND AA
      CALL SHSORT(AAA,KEY,N)
      CALL SHSORT(BBB,KEY,N)
      CALL SHSORT(SSS,KEY,N)
```

```
      CALL SHSORT(DDD,KEY,N)
      CALL SHSORT(EXPD,KEY,N)
      CALL SHSORT(DD,KEY,N)
      CALL SHSORT(EXPDD,KEY,N)
      WRITE(6,4999)
 4999 FORMAT('0','  INDEX ','     AAA    ','     BBB    ','     SSS     '
     1,'     DDD    ','     EXPD    ','     DD     ','    EXPDD   ')
      WRITE(6,1999) (INDEX(I),AAA(I),BBB(I),SSS(I),DDD(I),EXPD(I),
     1DD(I),EXPDD(I),I=1,N)
 1999 FORMAT('0',I8,7F12.8)
      WRITE(6,9999)  IX
 9999 FORMAT('0',9X,'IX =                ',I11)
      CALL NMPLOT(DDD,P,AREA,N,KAY)
      WRITE(6,1307)
 1307 FORMAT('0','TEST FOR NORMALITY DDD')
      CALL NMPLOT(EXPD,P,AREA,N,KAY)
      WRITE(6,1407)
 1407 FORMAT('0','TEST FOR NORMALITY EXPD')
      CALL NMPLOT(DD,P,AREA,N,KAY)
      WRITE(6,1607)
 1607 FORMAT('0','TEST FOR NORMALITY DD')
      CALL NMPLOT(EXPDD,P,AREA,N,KAY)
      WRITE(6,1707)
 1707 FORMAT('0','TEST FOR NORMALITY EXPDD')
 8000 CONTINUE
      END
      SUBROUTINE NMPLOT(XR,P,AREA,N,KAY)
      DIMENSION XR(N),P(N),AREA(N),KAY(N)
      SUMXR = 0.0
      SUMSQ = 0.0
      DO 1001 I = 1,N
      SUMXR = SUMXR + XR(I)
      SUMSQ = SUMSQ + XR(I)**2
 1001 CONTINUE
      XRBAR = SUMXR/FLOAT(N)
      VARXR = (SUMSQ - N*XRBAR**2)/FLOAT(N-1)
      DO 1011 I = 1,N
      XR(I) = (XR(I) - XRBAR)/SQRT(VARXR)
 1011 CONTINUE
      CALL SHSORT(XR,KAY,N)
      DO 1101 I = 1,N
      XX = XR(I)
      CALL NPOA(XX,5,XORD,XAREA,ERR)
      IF (XX.LT.0.0) GO TO 1201
      AREA(I) = XAREA
      GO TO 1101
 1201 AREA(I) = 1.0 - XAREA
 1101 CONTINUE
```

```
      DO 1301 I  = 1,N
      P(I) = FLOAT(I)/FLOAT(N)
 1301 CONTINUE
      WRITE (6,2222)
 2222 FORMAT('1','TEST FOR NORMALITY')
      CALL PLOTP(P,AREA,N,0)
      RETURN
      END
```

```
C ..............................................................................
C
C
C   4        WEIGHTED HUBER METHOD FOR THE MODEL Y=A*EXP(-B/T**C)*EXP(U)
C
C   A(K)        WEIGHTED HUBER ESTIMATE OF A
C   W(J)        WEIGHTS
C   TWO DATA CARDS ARE REQUIRED FOR SCALE PARAMATER
C
C
      DIMENSION T(20),X(20),Y(20),YY(20),RHO(30),U(20),WSHOT(20),A(100),
     1   B(100),C(100),A1(100),B1(100),KEY(100),Z(40),S(100)
      N = 10
      IT = 20
      M = 20
      PROB = 0.9
      IX = 6432943
      FTRUE = 1.0
      CTRUE = 0.0772
      BTRUE = 10.0
      CALL ERRSET(208,256,0,1,1,209)
      CALL OVFLOW
      READ(5,1777) (RHO(I),I=1,2)
 1777 FORMAT(F10.4)
      DO 8000 LL = 1,2
      WRITE(6,1666) RHO(LL)
 1666 FORMAT('1',9X,'RHO= ',F8.4)
C
      DO 9000 K = 1,N
      CALL SNORM(IX,Z,M)
      DO 3000 I = 1,M
 3001 WSHOT(I) = RHO(LL) * Z(I)
 3000 CONTINUE
      DO 3100 I = 1,M
      T(I) = I
      YY(I) = FTRUE * EXP(-BTRUE/T(I) + WSHOT(I))
      Y(I) = ALOG(YY(I))
      X(I) = -(1.0/T(I))
 3100 CONTINUE
C
      CALL ITRATE(Y,M,IT,AA,BB,CC,SS)
      A(K) = EXP(AA)
      B(K) = BB
      C(K) = CC
      S(K) = SS
 9000 CONTINUE
C
      WRITE(6,5000) RHO(LL)
```

```
5000 FORMAT('1',F10.4)
     CALL SHSORT(A,KEY,N)
     CALL SHSORT(B,KEY,N)
     CALL SHSORT(C,KEY,N)
     CALL SHSORT(S,KEY,N)
     WRITE(6,5100)
5100 FORMAT('0',7X,'A',9X,'B',9X,'C',9X,'S')
     WRITE(L,5200) (A(I),B(I),C(I),S(I))
5200 FORMAT(5F10.4)
8000 CONTINUE
     STOP
     END
     SUBROUTINE ITRATE(Y,M,IT,AA,BB,CC,SS)
     DIMENSION Y(1),A(30),B(30),C(30),S(30),ALOGJ(20),DJ(20),W(20)
     A(1) = 0.0
     B(1) = 10.0
     C(1) = 1.0
     S(1) = 0.25
     SUM7 = 0.0
     DO 1000 J = 1,M
     W(J) = 1.0
     DJ(J) = J
     ALOGJ(J) = ALOG(FLOAT(J))
      SUM7 = SUM7 + W(J)
1000 CONTINUE
     DO 9000 I = 1,IT
     WRITE(6,8888) A(I),B(I),C(I),S(I)
8888 FORMAT(4F9.4)
     SUM1 = 0.0
     SUM2 = 0.0
     SUM3 = 0.0
     SUM4 = 0.0
     SUM5 = 0.0
     SUM6 = 0.0
     SUM8 = 0.0
     SUM9 = 0.0
     DO 2000 J = 1,M
     SUM1 = SUM1 + (W(J)*PHI((Y(J)-A(I)+B(I)/DJ(J)**C(I))))/S(I)
     SUM2 = SUM2 + (W(J)*B(I)*(-ALOGJ(J))*DISCON((Y(J)-A(I)+B(I)/DJ(J)
    1     **C(I))/S(I)))/(S(I)*DJ(J)**C(I))
2000 CONTINUE
     C(I+1) = C(I) - SUM1/SUM2
     DO 3000 J = 1,M
     E = (Y(J)-A(I)+B(I)/DJ(J)**C(I+1))/S(I)
     SUM3 = SUM3 + (W(J)*PHI(E))/DJ(J)**C(I+1)
     SUM4 = SUM4 + (W(J)*DISCON(E))/(DJ(J)**(2*C(I+1))*S(I))
3000 CONTINUE
     B(I+1) = B(I) - SUM3/SUM4
```

```
      DO 4000 J = 1,M
      F = (Y(J)-A(I)+B(I+1)/DJ(J)**C(I+1))/S(I)
      SUM5 = SUM5 + (W(J)*ALOGJ(J)*PHI(F))/DJ(J)**C(I+1)
      SUM6 = SUM6 + (W(J)*ALOGJ(J)*DISCON(F))/(-S(I)*DJ(J)**C(I+1))
4000  CONTINUE
      A(I+1) = A(I) - SUM5/SUM6
      DO 5000 J = 1,M
       G = (Y(J)-A(I+1)+B(I+1)/DJ(J)**C(I+1))
      SUM9 = SUM9 + (W(J)*G**2*DISCON(G/S(I)))/S(I)**2
      SUM8 = SUM8 + (W(J)*G*PHI(G/S(I)))
5000  CONTINUE
      S(I+1) = S(I) - (S(I)*SUM7-SUM8)/(SUM7+SUM9)
      WRITE(6,6000) SUM1,SUM2,SUM3,SUM4,SUM5,SUM6,SUM8,SUM9
6000  FORMAT(44X,8F9.3)
9000  CONTINUE
      AA = A(IT+1)
      BB = B(IT+1)
      CC = C(IT+1)
      SS = S(IT+1)
      RETURN
      END
      FUNCTION PHI(X)
      IF(-2.0.GT.X) GO TO 100
      IF(2.0.LT.X) GO TO 200
      PHI = X
      GO TO 300
 100  PHI = -2.0
      GO TO 300
 200  PHI = 2.0
 300  RETURN
      END
      FUNCTION DISCON(X)
      IF(ABS(X).GT.2.0) GO TO 400
      DISCON = 1.0
      GO TO 500
 400  DISCON = 0.0
 500  RETURN
      END
```

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